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CONSTRUCTING MEANINGS AND UTILITIES WITHIN ALGEBRAIC TASKS

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The Purposeful Algebraic Activity project aims to explore the potential of spreadsheets in the introduction to algebra and algebraic thinking. We discuss two sub-themes within the project: tracing the development of pupils' construction of meaning for variable from arithmetic-based activity, through use of spreadsheets, and into formal algebra, and tracing the ways in which children construct utilities for algebraic activity. Our analysis of pupils' activity suggests that tasks which offer opportunities to construct different utilities may also be associated with the construction of different meanings for variable.

INTRODUCTION

The *Purposeful Algebraic Activity* project[1] aims to explore the potential of spreadsheets in the introduction to algebra and algebraic thinking, making links to both the learning and teaching of arithmetic and the development of traditional school algebra. In this paper, we discuss two sub-themes within the project: tracing the development of pupils' construction of meaning for variable from arithmetic-based activity, through use of spreadsheets, and into formal algebra, and tracing the ways in which children construct *utilities* for algebraic activity: that is, an understanding of why and how this is useful (Ainley and Pratt, 2002). We focus on the key algebraic idea of *generational activity* (Kieran, 1996): expressing relationships in a general way through the use of a variable.

Through the focused use of the spreadsheet environment, and carefully designed pedagogic tasks which are purposeful for pupils, the project aims to create opportunities for pupils to not only develop the technical skills of working with formal notation to express relationships, and conceptual understanding of this activity, but also to construct utilities for algebraic activity. We identify two potential utilities: generating many examples (so that patterns can be seen more clearly), and showing structure. Our analysis of pupils' activity suggests that tasks which offer opportunities to construct these different utilities may also be associated with the construction of different meanings for formal notation and variable.

MEANINGS FOR VARIABLE IN THE SPREADSHEET ENVIRONMENT

The different meanings for variable which may be constructed by learners in the early stages of algebra have been explored and reported by many researchers. Limited

[1] Funded by the Economic and Social Research Council

space does not allow a lengthy discussion but we draw on Ursini and Trigueros' (2001) recent categorization as a way of articulating one distinction which has become apparent within our analysis.

In the algebra-like notation of the spreadsheet, the cell reference is used ambiguously to name both the physical location of a cell in a column and row, and the information that the cell may contain. The spreadsheet thus offers a strong visual image of the cell as a container into which numbers can be placed. The meaning for variable which this image seems likely to support is that of a *placeholder for general number* (Ursini and Trigueros, 2001)**,** implying that pupils are able to:

> interpret a symbol as representing a general, indeterminate entity that can assume any value, and symbolise general statements, rules or methods (p. 336)

However, the image offered by the spreadsheet is ambiguous in another powerful way: when a formula is entered in a cell, it can be 'filled down' to operate on a range of cells in a column. The cell reference can then be seen as both specific (a particular number I may put in this cell) and general (all the values I may enter in this column). This image is likely to support the idea of variable as *a range of numbers in functional relationships*. Ursini and Trigueros (2001) associate this with (amongst others) the abilities to:

> determine the values of the dependent variable given the value of the independent one, and symbolise a functional relationship based on the analysis of the data of a problem. (p. 336-7)

Other features of the spreadsheet environment may offer opportunities for pupils to appreciate utilities of algebraic activity. The first is that its notation provides a 'language' which can mediate between pupils' natural language and formal algebraic notation (Sutherland, 1993). Meanings for 'spreadsheet language' develop during algebraic activity, alongside the use of natural language, to express ideas and relationships. The meaningful use of spreadsheet language may demonstrate the power of algebraic activity to allow the structure of a relationship to be easily seen. A second feature is that the use of spreadsheet notation has an immediate purpose in producing a result from which pupils can get meaningful feedback, such as a table of data which can be graphed. Pupils are thus able to appreciate the usefulness of algebraic activity to generate examples from which patterns can be seen.

These two features contrast sharply with more traditional approaches, where pupils may be required to translate their ideas into formal notation as the last stage of an activity. Here the purpose of expressing relationships in a formal notation may be unclear, and the only feedback accessible to the pupil may be the teacher's approval.

THE TEACHING PROGRAMME

Within the *Purposeful Algebraic Activity* Project we have developed a teaching programme of six tasks (used as three pairs) which incorporate different uses of the spreadsheet, and different algebraic ideas, within settings designed to have clear

purposes for pupils. The tasks have been developed by the project team, in collaboration with a group of primary and secondary teachers. The primary teachers trialled initial versions of the tasks with their pupils (aged 10-11 years), and this experience was fed into the development of a more polished set of materials to be used by the secondary teachers. These were used as part of the normal curriculum with five classes in the first year of secondary schooling (aged 11-12 years, covering a range of attainment, in two schools).

The data presented in this paper is taken from work on two tasks: *Hundred Square* (task 2 in the programme) and *Mobile Phones* (task 4).

DATA COLLECTION

The secondary classes in schools worked on each task for 2-3 lessons. Unfortunately, the time available for these lessons was limited by timetabling and curriculum constraints, and many classes did not have enough time to complete the tasks as they were originally designed. All lessons were observed by a researcher, and four sources of data were collected:

- researcher's field notes, which included observations of pupils' working
- audio-recordings from a radio microphone worn by the teacher
- video and screen recordings from a targeted pair of pupils in each lesson
- examples of pupils' written work and spreadsheet files.

The audio and video recordings were transcribed. The transcripts of the pairs of children were annotated with observations from the video and screen recordings, and examples of the pupils' files saved during the lessons. All sources of data were then coded to identify examples of different kinds of generational, transformational and meta-level activity (Kieran, 1996), including different meanings for variable, and use of natural language, spreadsheet and formal notation to generate expressions. Codings were cross-checked amongst the project team, and sub-themes emerged and developed during the coding process.

ACTIVITY IN THE *HUNDRED SQUARE* TASK

Hundred Square involves exploring patterns with a 100 square (created on the spreadsheet) by taking 3x3 cross-shapes from within the square, and comparing the sums of values on the two arms of the cross (see example in Fig. 1). To make the exploration easier, pupils are asked to set up a 'testing cross' on their spreadsheet, and use formulae to create the complete cross once the middle number is entered. They are asked to explain the patterns that they find in their results, and then set a final challenge: to design some new shapes on the Hundred Square with interesting patterns.

In Judith's class, most pupils managed to find some patterns in their results when adding the numbers on the horizontal and vertical arms of the cross, noticing that each gave the same total, and that this total was three times the number at the center of the cross. However, Louise was the only pupil prepared to try to explain this in the discussion at the start of the second lesson.

13	14	15	16
23	24	25	26
33	34	35	36
43	44	45	46
53	54	55	56

Figure 1

> Louise said that 16+10=26 and 16-10=6, explaining why the column total was three times the middle number. (*Field notes*)

With her partner Harriet, Louise was then comfortable with writing formulae to express this relationship on the spreadsheet, using *=B14-10* and *=B14+10* as the formulae above and below the middle cell (B14) in their 'testing cross'. Later they went on to make a new shape, a cross covering five rows and columns. They confidently used the cell reference of the center cell as the starting point for their formulae, but had difficulty in deciding how this related to the cells at the far ends of their cross.

Harriet	M13 "minus a hundred
Louise	No ten
Harriet	No minus, minus a hundred
Louise	Minus, no minus two and then minus ten (*Pair transcript*)

They continued in this way for some time before producing the cross shown in Figure 2. This was correct on the horizontal arm, but the numbers which would appear in the vertical arm of the cross (4, 12, 14, 16, 24) clearly could not appear in those positions in the Hundred Square. Louise explained to Harriet that the vertical arm was correct, on the basis of the symmetry of the formulae. This cross does, indeed, 'work' in so far as the sums for the two arms are equal, and these are 5 times the central number.

		=M13-10		
		=M13-2		
=M13-2	*=M13-1*	14	*=M13+1*	*=M13+2*
		=M13+2		
		=M13+10		

Figure 2

Louise and Harriet's conversation was all in terms of operations on the unknown central number in the cross, and they seem to be comfortable with using the cell reference as a placeholder for a general number. Other pupils tended to talk about particular values, but used similar arguments to explain the structure.

In Ann's class, Elizabeth and Shannon also worked confidently, producing appropriate formulae, and explaining their findings in general terms.

Elizabeth	We found out that the formula was seventy six times three equals the column and the row number (looks at Researcher) (.). Well not seventy six, the middle number times three equals the column and the row number …And that is because if you take ten from the, one of the column numbers (points to the bottom cell) and put it on the other column number (points to the top cell) then they both equal seventy six (*Pair transcript*)

In Graham's class, some pupils offered similar descriptions of why the row and column totals are the same, for example:

Pupil	Basically they cancel each other out because
Graham	What cancels each other out?
Pupil	Well the top, the number above would be minus ten and the number below would be plus ten so they'd cancel each other out and then that one would be minus one and that would be plus one (*Teacher transcript*)

Although the language here is imprecise, it is clear that the pupils were describing the structure of a general pattern, which might apply to any 'middle number', just as Elizabeth recognised that her pattern would not just apply to seventy six.

The examples offered here of pupils' activity in the *Hundred Square* task suggest strongly that they are constructing a meaning for variable as a placeholder for any 'middle number' in the cross, and using this generalized number to symbolize general rules, whether expressed as spreadsheet formulae or in natural language.

We suggest that they are also constructing a utility for the use of generalized expressions of relationships, however they are expressed: that of showing structure. In this task we see pupils moving between explanations which are strongly rooted in the arithmetic and physical structure of the Hundred Square, generalized descriptions, and spreadsheet formulae which reflect the symmetry of that structure, even when in the case of Harriet and Louise, the formulae they produce do not actually match that arithmetic structure.

ACTIVITY IN THE *MOBILE PHONES* TASK

In *Mobile Phones*, pupils are presented with information about two different tariffs offered by a mobile phone company, together with the calltime someone uses each month for half a year. They are asked to set up a spreadsheet so that they can investigate which tariff offers the best value.

Tariff	Monthly Rental	Calls
Tariff A	£12.95	20p per minute
Tariff B	£14.50	15p per minute

November	15 minutes
December	48 minutes
January	80 minutes
February	44 minutes
March	113 minutes
April	63 minutes

They are then asked to investigate three more possible tariffs, and write a brief guide to say which tariffs would be most suitable for different types of users.

At the beginning of the *Mobile Phones* task Harriet and Louise realized quite quickly what they needed to do to calculate the cost for November for tariff A.

Harriet	No 'cause that's "monthly (points to £12.95 on the table on the worksheet), you have to pay that anyway and then … that, times, where's the times (enters '=A2')
Louise	Why do you need "times?

Harriet	You need times ‘cause you need to that (points to 15 minutes) “times twenty (*Pair transcript*)

After some difficulties with syntax, Harriet typed in the formula correctly. Although she typed *A2* for the calltime for November, she actually talked about the calculation as ‘fifteen times nought point 2’. Louise realized that this would not achieve what they wanted.

Louise	No, no, Harriet listen. If you do fifteen times twenty (she means 0.20) then it’ll only work for fifteen, it won’t work for forty-eight (the calltime for December)
Harriet	No, no I put A, A”2 (points to cell A2 then looks directly at Louise as if checking that she understands) times twenty
Louise	It will only work for A2
Harriet	No it “won’t, it will work for A3 as well (points to A3) (*Pair transcript*)

In this exchange it is clear that both girls realized that they needed to give a general expression for the total cost, which would work for every month. Throughout their discussion, Harriet seemed to be using the specific number of minutes (15) and the cell reference (A2) interchangeably, suggesting that she is thinking of this quantity as a variable. Louise knew that using 15 would not give them an expression which would work for all the cases, but did not immediately recognize the power of using the cell reference. However, she was quickly convinced when they filled the formula down and she saw the costs for each month calculated.

In Ann’s class, Max and Peter also struggled with the syntax involved in entering amounts of money, and despite having a sense of the structure of the calculation, made several errors in working out the cost for November. When they began on the calculation for December (48 minutes) Peter was clearly getting more confused.

Peter	I don’t get you
Max	(faces Peter and points to the worksheet throughout) That’s like your monthly line rental right, so you have to do, you have to pay that every month …And then say you used forty-eight minutes, that’s twenty p per minute so you have to times them to there plus twelve pounds ninety-five
Peter	Go on then tell me (sits back in chair)
Max	↑Do a ↑”formula [loudly] (palms face upwards then holds hands against forehead) (*Pair transcript*)

Max’s frustration continued for some time as they struggled with the syntax, getting answers which they recognized were incorrect. Eventually they tried to attract the teacher’s attention, but before she arrived, Peter looked across at the screen of the pupils sitting next to them.

Peter	(looking at the computer of pupils sitting next to them) Cool (.) ‘ere we didn’t get that answer, actually we didn’t get that answer. … What formula did you get for that (points) ‘cause we couldn’t get a formula (..) A2

Max	Oh yeah it's supposed to be A2 times
Peter	(looks at Max) You "stupid "[thing] you
Max	Yes I am stupid, we put twenty times fifteen (*Pair transcript*)

Although they had used cell references confidently in earlier tasks in the programme, Max and Peter had not immediately recognized the power of using them here. Max's explanation suggests that his attempts to enter a formula were based on a clear image of the general structure of the calculation. Once he realized that he had been 'stupid', Max insisted that he could quickly complete the rest of the task, saying '*I can get all these answers down in two seconds*'.

Again in this task we see pupils moving from arithmetic calculations, to generalized calculations of the dependent variable (the total monthly cost), which are often expressed in informal language. These functional relationships are then formally expressed as spreadsheet formulae which are replicated to generate sets of data, suggesting that pupils are constructing meanings for variable as a range of values.

Towards the end of the lesson, Peter and Max discussed their progress with the researcher.

Researcher	How you getting on?
Peter	Okay
Max	We've now realised what we're doing (laughs)
Researcher	Why did you decide to write a formula?
Peter	'Cause it's easier
Max	'Cause it's quicker (*Pair transcript*)

Like Harriet and Louise, Peter and Max could now clearly see the utility of expressing the calculation of monthly costs in a general way, so that they could generate a lot of data for each tariff quickly. Pupils across all five classes went on to use line graphs to compare Tariffs A and B, and most were able to make some links between the way in which these costs changed as calltime increased, and the cross-over points on the graph. Comparing the costs of further tariffs proved to be more challenging, and for all but a few pupils, time ran out before they were able to complete their users' guide to all the tariffs. However, many did make some recommendations like the following in their reports, indicating that they had appreciated the value of comparing sets of data rather than individual values.

> We have found out that if you spend a lot of time on your phone (1 and a half hours to 2 hours) you should go with Tariff C ... If you spend about (30 minutes to 1 and a half hours) on your phone then you should go with Tariff B ... if you spend a little amount of time on your phone (0 to 30 minutes) you should go with Tariff A.
>
> Tariff A is cheeper (*sic*) up to 30 minutes but when you get to 31 minutes Tariff A and B are equal, after that Tariff B is cheeper. (*Written reports*)

DISCUSSION

The snapshots of data presented here give an indication of the complex interactions between different elements in shaping the ways in which meanings are constructed in what may superficially appear to be similar activity on the part of pupils. In both tasks, pupils have a problem to solve involving generational activity, which takes the form of writing spreadsheet formulae. In the examples presented in this paper, we see the overall purpose of the task (understanding and explaining an intriguing pattern in order to design a new one, and comparing patterns of data produced by similar functions) as crucial in influencing the way in which the spreadsheet is used, and thus the meanings and utilities which may be constructed.

The teaching programme was designed around the ideas of different kinds of algebraic activity (generational, transformational and meta-level), opportunities for exploiting different features of the spreadsheet environment, and possibilities for pupils to move between arithmetic and algebraic structures, using natural language and informal notations, spreadsheet notation and formal algebraic notation. The sequence of tasks in the programme aims to combine these elements with different foci, in progressively more complex algebraic activity. However, as we have observed the use of the tasks by different teachers it has become apparent that subtle changes in emphasis by the teacher can lead to changes in the way that the purpose of the task is perceived. For example, an attempt to simplify the introduction to *Mobile Phones* may lead pupils to focus on calculating the cost for each month separately, rather than seeing the functional relationship between minutes used and total cost, so that the need for constructing a general expression using a variable is lost. As our longitudinal analysis of the teaching programme data continues, the ways in which the focus of attention within tasks may be shaped by teachers' and pupils' perceptions of purpose, and of the role of the spreadsheet, will be a significant theme.

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PROPORTIONAL REASONING OF QUASI-ILLITERATE ADULTS

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The main purpose of this paper is to describe the answers given by adults without primary schooling to different ratio- and rate-comparison tasks. The framework and the analysed data are part of an ongoing research, in which the responses of subjects of different ages and schoolings are studied. The behaviour of quasi-illiterate adults could throw some light on the effect of school on proportional reasoning in normal conditions; evidence will be shown regarding the similarities of their phenomenological behaviour with the one of people with regular schooling, especially the influence of number structure and context upon proportional reasoning.

Part of an ongoing research on the strategies used by subjects of different ages and schoolings when faced to different kinds of ratio comparison tasks is reported in this paper. In the part here conveyed, we are concerned with the following question: Is proportional reasoning something developed exclusively at school, or does daily life provide the means to it? This question is difficult to answer in developed countries where all the population has a minimum of several years of schooling in which proportionality is taught. However, in Mexico about 20% of the adult population has none or very little schooling, which of course is nothing to be proud or glad about, but allows us to consider illiterate adults as epistemic subjects who could lead us to a tentative answer to our question.

PREVIOUS WORK: A FRAMEWORK

As stated above, we are investigating the strategies used by different subjects when faced to different kinds of ratio comparison tasks. In Alatorre and Figueras (2003) we explained what it is meant by "different kinds of ratio comparison tasks" and described the interview protocol used in the experimental part of the research. Some of the categories for classifying the questions, as well as the categories (strategies) for interpreting the answers, stem from a framework presented in Alatorre (2002). A succinct summary of both papers will be sketched here; the reader is referred to them for a more complete account.

Among the problems calling for proportional reasoning, those in which the task is a comparison of ratios or of rates can be classified according to three issues: context, quantity type, and numerical structure. Table 1, taken from Alatorre and Figueras (2003), proposes a joint classification according to the first two; it blends together the classifications proposed by several authors (Freudenthal, 1983; Tourniaire and Pulos, 1985; Lesh, Post and Behr, 1988; Schwartz, 1988; Lamon, 1993).

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Rate problems: couples of expositions		Intensive quantity surging from two quantities: both discrete, both continuous, or one of each type
Part-part-whole problems: couples of compositions	Mixture	One quantity, discrete or continuous
	Probability	One quantity, discrete or continuous
Geometrical problems: couples of ∑-constructs		Two continuous quantities

Table 1: Taxonomy of ratio comparison tasks according to context and quantity types

The third issue is the numerical structure. Before describing it, the notation used in this paper will be presented. In a ratio or rate comparison there are always four numbers and two "objects" (1 and 2) involved. In each object there is an antecedent "a" and a consequent "c", and thus the four numbers may be written in an array, which is an expression of the form $(a_1,c_1)(a_2,c_2)$. Also of interest may be the totals t=a+c, the differences d=a–c, and the part-whole quotients p=a/t. Alatorre's (2002) proposition is a classification of all arrays in 86 different situations according to 17 different "combinations" –successions of results when an order relationship is established in the array between the pairs of numbers t, a, c, d, and p–, and 17 different "locations" –non-ordered pairs of the following alternatives for both quotients of the array: *n:* nothing (p=0); *l:* lose (0<p<½); *d:* draw (p=½); *w:* win (½<p<1); and *u:* unit (p=1)–. The 86 situations can be grouped in six difficulty levels, labelled I to VI. Because of space limitations, for the purpose of this paper only three difficulty levels will be used: L1 (65 situations belonging to levels I, II, and III), L2 (3 situations belonging to level IV), and L3 (18 situations belonging to levels V and VI). The description of L1, L2, and L3 will follow the next paragraph.

In the previous paragraphs a description of the classification of ratio-comparison problems was given. Here follows a classification of the strategies used by subjects in their answers to such problems. Alatorre's (2002) framework, as presented in Alatorre and Figueras (2003), is to be used. Strategies can be simple or composed; in turn, simple strategies can be centrations or relations. Centrations can be on the totals CT, on the antecedents CA, or on the consequents CC. Relations, either "within" or "between", can be order relations RO (when an order relationship is established among a and c elements of each object and the results are compared), or subtractive relations RS (additive strategies), or proportionality relations RP. (For the purposes of this part of the research, RP relations were decomposed in several categories, which will be described further on). Composed strategies of two or more simple ones can be conjunctions $X\&Y$ (X and Y dominate), exclusions $X\neg Y$ (X dominates), compensations $X*Y$ (X dominates), or counterweights $X\perp Y$ (neither dominates). Some examples will be given further on. Strategies may be labelled as correct, sometimes depending on the situation (combination and location) in which they are used. Correct strategies are RP in all situations; RO in *wl*, *wd*, or *dl* locations; CA in locations with *n*; CC in locations with *u*; and, in some situations, some composed strategies that can be considered as theorems in action (see e.g. Vergnaud, 1981).

The three difficulty levels mentioned before refer to which correct strategies may be applied. Grouped in L1 are all the situations where, in addition to RP, all other correct strategies may be used. In L2 and L3 only RP can be used; the difference among them is that L2 consists of situations of proportionality (both ratios or rates are the same), and L3 consists of situations of non-proportionality.

METHODOLOGY

For the part of the research reported in this paper, a case study was conducted with six adults. They were students at a Centre for Adult Education in Mexico City; Reyna, Zoraida, and Ubaldo (aged 17, 49, 65) were learning to read and write, and Luisa, Dalia, and Toñita (aged 24, 25, 51) were studying a correlative primary school. All had less than the equivalent to four years of schooling. Four worked as housemaids, Toñita owned a small shop, and Ubaldo was a builder. They were interviewed for a time between 60 and 90 minutes, and the sessions were videotaped.

During the interviews, subjects were posed several questions in each of eight sorts of problems, which were Rate problems and both kinds of Part-part-whole problems (in this research Geometrical problems are not dealt with). Table 2 describes them.

CONTEXT				Objects	Antecedent	Consequent	Question
Rate		N	Notebook problem	Stores	Notebooks (d)	Coins (d)	In which store are the notebooks cheaper?
		B	Blocks problem	Walking girls	Blocks (x)	Time (x)	Which of the two girls walks faster?
		Y	Yards problem	School-yards	Children (d)	Yard squares (x)	Which of the yards is more cramped with children?
		L	Lemonade problem	Jars	Lemons (d)	Cups with sug-ared water (x)	In which jar is the lemonade's taste stronger?
Part-part-whole	Mixture	E	Exams problem	Exams	Correct answers (d)	Incorrect answers (d)	In which exam did the student do better?
		J	Juice problem	Jars	Concentrate glasses (x)	Water glasses (x)	In which jar does the mixture taste stronger?
	Probability	M	Marbles problem	Bottles	Blue marbles (d)	Yellow marbles (d)	In which bottle is it more likely to get a blue marble?
		S	Spinner problem	Spinners	Blue sectors (x)	Yellow sectors (x)	In which spinner is a blue sector more likely to be marked?

Table 2. The eight problems of the protocol (d=discrete; x=continuous)

Each of the problems was posed in different questions according to numerical structure. Fifteen such questions were designed, five in each of the difficulty levels; they are displayed in Table 3. All the problems may be posed in each of 15 questions, except for question 7, which has no sense in problems B or Y (see table 2). Questions 1 to 10 were posed to all subjects; questions 11 to 15 only to those who used RP in

Level	Question numbers and their arrays (a_1, c_1) (a_2, c_2)				
L1	1 (2, 3)(2, 3),	2 (1, 4)(3, 2),	3 (2, 3)(2, 3),	6 (2, 2)(3, 2),	7 (3, 3)(2, 0)
L2	5 (3, 3)(1, 1),	8 (2, 1)(4, 2),	10 (3, 6)(1, 2),	12 (4, 6)(2, 3),	15 (8, 4)(4, 2)
L3	4 (2, 1)(3, 2),	9 (2, 5)(1, 3),	11 (5, 2)(7, 3),	13 (3, 2)(5, 3),	14 (2, 4)(3, 5)

Table 3. Numerical structure of the fifteen questions

previous questions of the same problem. The problems were presented to subjects in a graphical form, which can be consulted in Alatorre and Figueras (2003); Figure 1 is an example. The problems were posed in this order: M, N, J, Y, S, L, E, B (see table 2). Within each one, 10 to 15 questions were posed. In each of them, the subjects were asked to make a decision (left side, right side, or "it is the same") and to justify it.

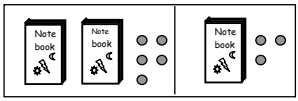

Figure 1. Question N-9

RESULTS AND ANALYSIS

A total of 513 answers was obtained; 424 (83%) of them were classified using the strategies system described above, and the rest either consisted of a decision without a justification (35), or had a justification that was only a description (27), or consisted of solution mechanisms different from the strategies described before (27). How these non-classifiable answers were handled will be described below.

Two phases of analysis were undertaken: first a quantitative, then a qualitative one. In order to make a quantitative analysis possible, one point was given to all correct strategies, and ½ point was given to answers that could be incomplete expressions of correct theorems in actions. Also, ½ point was given to all non-classifiable answers that fulfilled the following conditions: correct decision and either no mechanism or a mechanism that could eventually become correct (such as arithmetic or geometric approximations). Then, for each subject a score was obtained for each group of context and difficulty level, and expressed as a percentage of the questions answered by the subject in that group. The results are shown in Figure 2.

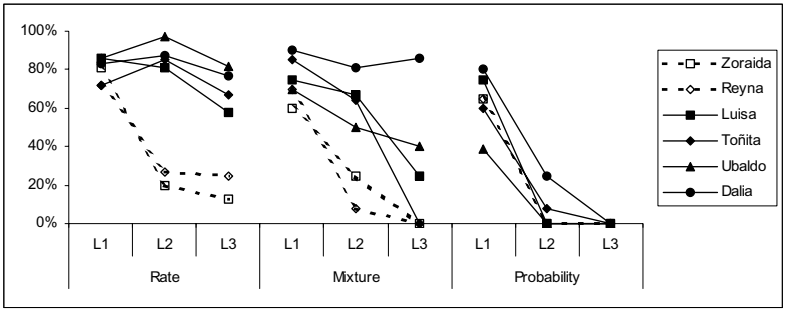

Figure 2. Subjects' performance

As Figure 2 suggests, the six subjects can be divided in two groups. Both groups obtained their best results in Rate and their worse in Probability problems, but with different behaviour. Subjects of the first group (Zoraida and Reyna, dotted lines) only answered fairly well the L1 questions of all contexts and dropped their performances in levels L2 and L3. The latter proved to be very difficult for them; their performance averaged 19% in Rate problems and was null elsewhere.

The remaining subjects had also their lowest results in L3 of all contexts, but only in Probability problems was it null for all of them, and elsewhere the decrease was not as marked as in the first group. Compared to the first group, subjects of the second one had a tendency to obtain better results in all L1, and significantly higher performances in L2. All of the subjects except Luisa even had better results in L2 than in L1 in the Rate problems, showing that they could recognise and appropriately solve situations of proportionality.

For the qualitative phase of analysis, the strategies used by subjects were studied. All sorts of strategies were used; here are some examples of centrations, RO and RS:

Reyna: I choose the right side, because there're more children and the space is smaller (question Y-2, correct composed strategy CA & CC).

Ubaldo: She did better on the left side because of the correct ones, although she has six wrong answers (question E-10, incorrect composed strategy CA ¬ CC).

Reyna: I choose the right side, because it has more concentrate than water, and in the other one they're the same (question J-6, correct simple strategy RO).

Luisa: It's the same, because in both there are more blue marbles and fewer yellow ones (question M-9, incorrect simple strategy RO).

Zoraida: The right side, because there're two more lemons than cups, and on the left side there's only one more (question L-8, incorrect simple strategy RS).

As for the correct proportionality relations, it became necessary to focus on the different kinds of RP. Four kinds were identified, which are described below:

RPM ("Multiples"): The subject realizes that there are multiples among the numbers of the array, either within an object or between objects. Example:

Dalia: It's the same: The left side is half as much as the right side (question E-8).

RPG ("Groupings"): The subject uses groups of specific amounts of antecedents and consequents, identifies them as appearing once or several times in each object, and compares the remaining elements (if any). Example:

Toñita: I choose the right side, because on the left side there's one and a half lemon for a cup, on the right side one and a half for one, one and a half for another one, and one and a half for another one. And there's half a lemon more (question L-13).

RPE ("Equalizing"): The subject executes a physical or a mental action of multiplying or dividing one of the objects by a certain quantity and by doing so either

equalizes both objects, or he/she equalizes the antecedents (or consequents), which permits the comparison of the consequents (or antecedents). Example:

> Ubaldo: I choose the left side, because in one more minute the girl would walk four blocks, and the one on the right [only] walks three blocks in those two minutes (question B-4).

RPR ("Rate or ratio comparison"): The subject calculates in each object the rate or the ratio a:c, the part-part quotient a/c or the part-whole quotient a/t. This is generally accomplished through the calculation of the unity value. Then he/she compares both results. Example:

> Luisa: I choose the left side, because on the right a notebook costs three coins, and on the left it would cost less than three coins, about two and a half coins (question N-9, see Figure 1).

The qualitative analysis is based on two considerations: whether the correct strategies were RP (and, in that case, what kind of RP), and the classification of incorrect answers. As with the quantitative analysis, the same two groups can be identified.

In the first group, Zoraida and Reyna's behaviour was characterized by the use both of centrations, either in simple or in composed strategies, and order relations. These are strategies that may lead to correct answers in L1, but necessarily lead to incorrect ones in L2 and L3. In the case of these two subjects, these strategies account for all of the correct answers in L1 and, mainly centrations, for most of the incorrect ones in L2 and L3. The few exceptions are some additive strategies and, in the Spinner problem, a mechanism of choosing the spinner with a bigger chunk of blue colour, such as the left side in Figure 3. In levels L2 and L3 each of the subjects had six scarce correct RP strategies, in the Notebooks and Lemonade problems, and once each in the Blocks and the Juice problems. These few attempts at proportional reasoning strategies were mainly of the RPM kind in L2 proportionality situations. Reyna displayed as well a couple of RPEs and Zoraida a couple of RPRs.

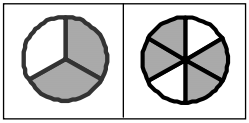

Figure 3. Question S-8

In the second group, the rest of the subjects showed a much richer behaviour, characterized by what seems to be a quest for the easiest correct strategy. In L1, the four subjects profusely used the simple and correct RO relations and theorems in action; however, all of them used at least once a RPR strategy. All four strategies leading to the answer "it is the same" in L2 were used: mostly RPR, but also RPG, RPE, and RPM (in this order); and all but RPM (in the same order) were used to choose one of the objects in L3. Save for Ubaldo, who never used RPM, all four subjects used all four RP strategies, in the same frequency order. This was especially notorious in the Rate problems, where RPR accounts for 64% and RPG for 27% of all RPs. As for the distribution among contexts, all four subjects used RP strategies in the four Rate problems, with small variations (RP were more frequently used in the Lemonade problem and less frequently used in the Yard problem), but used them

much more sparingly in Part-part-whole problems: Two of the subjects (Ubaldo and Luisa) used them only in the Juice problem, one (Toñita) used them as well twice in the Spinner problem, and only one (Dalia) used them in the Exams problem. The use by these subjects of RP strategies in Probability problems was very scarce (Spinners) or null (Marbles).

An analysis of the incorrect answers of the four subjects of the second group leads to the following. In the Rate problems, only one of the subjects (Luisa) used centrations and RO relations incorrectly in L2 and L3; all other incorrect answers were due to either additive relations RS or incorrect attempts at some of the RP strategies, mainly with arithmetic mistakes. This was also the case in the Juice problem. In the Exams problem all four subjects incorrectly used centrations, and some of them used incorrect RO and RS relations. The abundant incorrect answers in both Probability problems range from centrations to RO relations to the mechanism of big chunks described above but also the inverse mechanism: choosing the side where the blue sectors are more scattered, such as the right side in Figure 3. There were also some mechanisms due to misconceptions of randomness, such as the following:

Ubaldo: It's the same. If it's my luck, I win. I win 'cause I win. If not, I don't win; even with twenty blue marbles and one yellow one, I don't (question M-4).

CONCLUSIONS

Both phases of the analysis lead to similar interpretations. Some of the subjects with little or no schooling approach the rate- and ratio-comparison problems in ways similar to that reported in the literature about young children (e.g. Noelting, 1980): their first choice being centrations, they only succeed where centrations lead to the correct answer, and fail elsewhere. This is to say, their performance depends deeply on the numerical structure of the questions, and they succeed where it allows non-proportional reasoning. However, they do occasionally produce one form of proportional reasoning, mostly in proportionality situations.

Another group of subjects does produce an assortment of forms of proportional reasoning. They succeed in the most easy questions mainly by using correct strategies different from the proportionality relations, but they also may succeed in questions where the only way to reach a correct answer is with the use of proportional reasoning. Here they used different kinds of strategies, apparently in a search for the easiest one. The performance of these subjects depends on the context: it is fairly good in all Rate problems, but decreases in Part-part-whole problems. The relative success in Mixture problems is due mainly to success with the Juice problem, since the other Mixture problem, Exams, had a very low success rate, probably due to the fact that exams are alien to the experience of these subjects, who are only beginning their schooling. This corroborates that familiarity with the problem is one important factor for success in proportionality problems (Tourniaire and Pulos, 1985; Lamon, 1993). Both Probability problems had also very low success rates, which may be due to the combined effect of lack of familiarity and the difficulties of randomness.

This study confirms some of the findings reported in the literature of proportional reasoning (Tourniaire and Pulos, 1985): that there are important effects of numerical structure and of context, that Rate problems are easier than Part-part-whole ones, and that familiarity with the problem is crucial. Therefore, these results seem to be independent of the schooling of the subjects considered.

The responses of the quasi-illiterate subjects who participated in this study resemble those of some university students (Alatorre, 2000): Some use non-proportional strategies and get entangled in number structures which are more complex than the easiest ones, whereas some are able to use proportional strategies and surmount the numerical difficulties. The latter obtain much better results than the former, although their performance is context-dependent.

The literature on the subject has long ago demonstrated that schooling is not a sufficient condition to reach an appropriate proportional reasoning. The fact that daily life has provided some of the quasi-illiterate subjects with a fairly good performance, at least in Rate and some Part-part-whole contexts, seems to suggest that schooling might also not be a necessary condition for proportional reasoning.

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USES OF EXAMPLE OBJECTS IN PROVING

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This paper builds upon discussions of the importance of semantic or intuitive work in proving by identifying three ways in which experienced mathematicians use example objects in constructing and evaluating proofs. It observes that students often do not use objects in these ways, and discusses the pedagogical question of how we might teach students to be more effective in proving by designing instruction to focus their attention on relevant objects. Data are drawn from interviews with five mathematicians experienced in teaching an introductory proofs course.

INTRODUCTION

It is well-recognized that students sometimes attempt to prove a general statement by empirical means, checking a number of examples to give evidence of its truth, rather searching for a deductive proof (e.g. Harel & Sowder, 1998). This is considered to be an inappropriate approach, and students are warned not to "prove by example". On the other hand, it is also noted that semantic or intuitive considerations can be very important in the work of successful mathematicians (e.g. Thurston, 1995). This paper builds on these considerations by offering more precise characterizations of the ways in which successful mathematicians use example objects to aid in proof construction and evaluation.

RESEARCH CONTEXT

The characterizations to be given below are derived from analysis of interviews with mathematicians experienced in teaching a course called "Introduction to Mathematical Reasoning", which is designed to provide students with a grounding in proving before they take courses in real analysis and abstract algebra. This is taught at a large state university in the USA. Classes typically have between 20 and 25 students, so that the professors are in relatively close contact with individuals and become familiar with their work during the 14-week course.

Five participants were involved in this exploratory study, which set out to address a gap in mathematics education research on proof and proving by drawing on the experience held by mathematicians who teach such courses, and seeking to formalize this into knowledge that can be more readily discussed and applied. Each participant was interviewed up to three times during a year. The first interview asked the participant to describe their experience in teaching this

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course, to give their views on important things that students should learn, and to describe common student mistakes and misunderstandings and their pedagogical strategies.

These interviews were transcribed and analyzed following Glaser (1992). First, conceptual descriptions were added to the transcript, and summarized in memos. Concurrently, further memos were made about questions arising from this data, typically of a need for clarification, or of a possible theoretical link between comments. At this stage the analysis rotated regularly from one participants' interview to another, in order to facilitate synthesis of the ideas raised and to avoid becoming focused on the opinions of a single participant. Next, the memos were sorted according to their main substantive and/or theoretical content, producing a system of categories. Subsequent interviews asked more specific questions designed provide increasing saturation of the categories.

One outcome of this analysis was the identification of three uses of example mathematical objects in the mathematicians' reasoning, with a frequent lack of such use on the part of students. These are: (1) understanding a statement, (2) generating an argument, and (3) checking an argument. They are described and illustrated below.

UNDERSTANDING A STATEMENT

Our first point is that mathematicians view the instantiation of objects as important in reaching a meaningful understanding of a mathematical statement. In these excerpts, Professor 1 remarks upon this as a natural first step in understanding a definition.

> P1: …So one of the things, again, that's second nature to me but it's not to them [the students], is that if I see a definition, I immediately instantiate it. You know, just try some examples of this definition, and try to fit it in.

> P1: …what happens is…that you describe a new definition, you say "let f be a function, let x be a real number, we say that…" and then "some relationship between f and x holds if…blah, blah, blah." So then what they have to do, they have to realize that this definition only makes sense in the context of, I have to have a function in mind and I have to have a [number] in mind…

He notes, however, that such instantiation in response to a definition is not typical behavior for students in his classes.

> P1: And what they'll do is typically if you have a sequence, you know, if I have a sequence definition to use in the rest of the problem, and they don't understand the definition, they'll just skip that sentence and go on. I will – they will come in for help on a problem, and five or ten minutes into the discussion I'll realize that, that they never bothered to process this particular definition. They have no idea what this means.

In response to this phenomenon he has invented task sequences involving the construction of example objects that satisfy various combinations of properties associated with certain definitions.

> P1: So, what I've been trying to do is to have these exercises where the whole purpose of the exercise is just for them to process a mathematical definition. […] I have one where I, where I just define what it means for a – well, what a partition of a set means. I define it formally, so it has these two conditions, a collection of subsets, such that the empty set is not one of the subsets, for every element of the underlying set there is a subset that contains it, for any two sets in the partition the intersection is empty […] And then I just ask okay, construct three examples of a partition on the set {1,2,3,4,5}. And then, okay, construct an example of a collection of sets on {1,2,3,4,5} which satisfies the first two properties but not the third. The first and the third properties but not the second, the second and the third properties but not the first.

This task sequence resembles those suggested by Watson and Mason (2002), who report that requests for examples satisfying various constraints can encourage students to extend their thinking beyond "typical" examples. Such example generation is also recommended by Dahlberg and Housman (1997) on the strength of its effectiveness as a learning strategy when faced with a new definition.

GENERATING AN ARGUMENT

In the second use of example objects the mathematician either builds objects or instantiates known ones with the goal of generating a proof. The professors spoke of one way in which this might be achieved directly, and another less obvious heuristic that uses an informal version of the indirect argumentation used in proof by contradiction. The direct use involves trying to show that a result is true in a specific case, in the hope that the same argument or manipulations will work in general.

> P1: It's just to get them…if they have to prove, "for all *n*, something", when they come to the induction step, and the induction step is not completely trivial, so it actually involves…actually think about it and you have to come up with an idea. And so, well how do you go about finding this idea? And I, I try to convey to them, that the first thing you do, is that…suppose I'm trying to prove it for *n* equals 10, how can I use that it's true for *n* equals 9? So, do something very specific, do a concrete example and try to reason from that.

> P5: See if you can get from 2 to 3, if you can't get from *n* to *n* plus 1.

The second, indirect case reveals a less obvious way of thinking about proving universal statements. In the excerpt below, Professor 1 talks about generating a proof for such a statement by searching for a reason why one could not find a counterexample to this statement.

> P1: …the way I often think about a proof is that, you know you imagine this as, try to beat this. Meaning, try to find a counterexample. […] If you think about, if you think

about the reason why you were failing to find a counterexample, okay, then, that sometimes gives you a clue, to why the thing is true.

When the interviewer commented that it seemed non-obvious that one would try to prove a universal statement by thinking about why there could never be cases for which it did not hold, he remarked that in fact he considered this a natural approach, and gave the following explanation.

P1: The natural…the sort of natural thing that our brains can do, is sort of build examples and check them. Okay, and…all you, you know if one thinks of universal statements as saying that it's really a statement of impossibility, it's the negation, right? It's a statement that you can't…do something. […] And the way you understand that you can't do it is by thinking about doing it.

Professor 2 describes this indirect strategy very concisely, and in doing so highlights its relationship to a straightforward way of proving an existential statement.

P2: …if it's an existential statement I look to see whether I can produce an example. And if it's a universal statement I probably try to show that I can't find a counterexample.

Professor 1's comments resemble the ideas of "mental models" theories in which human beings generate and evaluate deductions by instantiating a model of the situation under consideration, evaluating a statement relative to that model, then varying the model in a search for counterexamples (Johnson-Laird & Byrne, 1991). The difference here is that one is not looking for a counterexample, but for *a reason why one cannot build one*, which will then form the basis for an argument.

CHECKING AN ARGUMENT

The third way in which professors routinely use objects in proving is in checking the correctness of individual deductions. Professor 1 describes this process in the abstract as follows

P1: …there is a locality principle about proofs, about every proof, and that is that we somehow recognize that even though you're proving some very specific thing, that there are portions in the argument. Each portion in the argument is actually doing something more general. […] And therefore I can do a local check on this part of the argument by thinking about that more general situation and doing, examples within that more general situation […] so that's one thing that, you know, is just completely second nature to me and to most mathematicians, is that you're constantly doing those kinds of checks.

A good illustration of this process is provided by Professor 3 in the following comments on a student proof attempt.

P3: For instance, problem: "express the number 30 as the difference of two squares. Or show that it cannot be done". Answer: "It cannot be done because 30 is divisible by 6 and a number that is divisible by 6 cannot be written as the difference of two squares." Well, 12 is 16 minus 4. Ah…take any number that's a difference of two squares,

> multiply it by 36, you'll get a number that's the difference of two squares and is divisible by 6...

In this case, although the proof is about the number 30, a general claim is made about numbers that are divisible by 6, and it is this claim that is shown to be incorrect by considering examples in the "general situation" of numbers that are divisible by 6. Once again, the study indicated that students appear not to engage in this process to a degree that their professors would like. In this excerpt, professor 3 expresses frustration at the fact that students regularly write statements that are "obviously wrong", in the sense that they could readily be refuted using such checks.

> P3: …I don't have a clue as to…what gets them…to, to say things like that. In other words I would say, things that are obviously false. To a normal person with a little bit of mathematical education it would seem obvious that you could never say such a thing because it's so obvious that it's false. Take any example that you want, you see clearly that it's false.

DISCUSSION

Summary

This paper contributes to our understanding of the semantic aspects of proving by identifying three specific ways in which thinking about example objects can assist in this process:

1. Instantiating examples in order to understand a statement or definition.
2. Generating an argument for a universal statement, by (directly) arguing about or manipulating a specific example and translating this to a general case or (indirectly) trying to construct a counterexample and attending to why this is impossible.
3. Considering possible counterexamples to general claims in a proof.

Each of these involves the consideration of example objects in a crucial way, as opposed to algebraic manipulations or deductions based only on the form of a statement. Each is also considered natural and even "second nature" by the mathematicians who took part in this study; they regularly commented upon their surprise when students made errors that could have been avoided by their use.

There are at least two possible reasons why students may not use example objects effectively in the construction and evaluation of proofs. One is that a key difference between novice mathematics students and their teachers is simply that the teachers have access to a great deal more experience with examples (Moore, 1994). A second is that students may not be accustomed to thinking in about the objects to which statements apply, instead thinking of mathematics (including proof) as a procedural enterprise in which algebraic statements are

laid out and manipulated according to certain rules or a standard format (Hoyles, 1997).

In either case, it seems that it would be helpful for students to improve their knowledge of example objects. However, experience in itself may be insufficient to support successful proof attempts; we do not wish students to offer examples in place of deductive proof. I suggest that the specific uses of example objects outlined above can help us to think about teaching students to use such objects more effectively.

Pedagogy: understanding a statement

Dahlberg and Housman (1997), Mason and Watson (2002) and Professor 1 all suggest setting tasks that require students to generate examples with given combinations of properties. Another possible design involves providing both objects and properties and asking students to decide which properties apply to which objects. I have used such a task in an introductory real analysis class. Students were presented with this list of subsets of the reals, and definitions of the given topological properties:

$\varnothing$, $\mathbf{N}$, $\mathbf{Q}$, $\mathbf{R}$, $\{0,1\}$, $(0,1)$, $[0,1]$, $[0,1)$, $(0,\infty)$, $\{1/n | n \in \mathbf{N}\}$, the Cantor set

Open, limit point, isolated point, closed, bounded, compact

They were asked to work in pairs and decide (without proof) which sets (or which points) satisfied which properties. Such a task design does not ask for example construction, but may be useful in cases where the objects to be considered are unfamiliar or where one wishes students to engage with non-standard examples. Certainly it goes beyond the typical lecture or book presentation of one or two "standard" examples for each definition, thus arguably encouraging the idea that we can ask about the applicability of properties more widely, and discouraging reliance on "prototypical" concept images (Vinner, 1991).

Pegagogy: generating an argument

In this study, the professors' descriptions of the indirect heuristic seem particularly interesting, since mathematicians generally find it difficult to describe the origin of the "key ideas" (in the sense of Raman, 2003) that lead to a proof. This heuristic is one way of systematically seeking such ideas, and could be articulated as part of a broad strategy for proving universal statements in which one first tries to prove the statement directly, and, if this fails, tries instead to construct counterexamples to the statement, articulating what prevents this from being accomplished.

Of course, it is often not easy to clearly articulate a mathematical claim. However, the heuristic seems no less teachable than that of trying particular examples in the hope of finding a generalizable argument. Its initial

introduction may be facilitated by experience with certain tasks – Antonini suggests that questions of the form "Given A, what can you deduce?" are likely to lead to indirect arguments (Antonini, 2003). Once in place its regular use by a professor may help students to overcome the feeling that they do not know how to proceed when faced with a statement that seems "obviously true" or otherwise difficult to prove.

Pedagogy: checking an argument

This study suggested that the idea of checking an argument by considering possible counterexamples is already part of the culture of the participants' classrooms. They informally model the process when explaining their own reasoning, and give tasks that ask students to identify false statements and give counterexamples. Here I would like to suggest that the use of these strategies could be strengthened in two ways.

First, by highlighting the fact that counterexample-production tasks have the same structure as that of making a "local check" of a deduction in a proof, but that this is often disguised by the fact that instead of one line saying "For all A, B" (or "If A then B"), we have two lines of the form:

A.

Therefore/so/hence B.

Although this seems a trivial distinction to a mathematician, it may not to a student who is struggling to coordinate their understanding of the form and content of an argument.

Second, by giving specific consideration to how to decide which objects to check. The subtleties involved in this process are highlighted by the following example of a student proof attempt given by professor 3.

> P3: …for example, I can show you a homework problem in which a student is trying to prove that the number 1007 is prime, and he said "well 7 is prime, and adding 1000 doesn't change anything". End of proof.

Faced with such a claim, the mathematician infers a general statement about adding primes to other numbers, and searches for a counterexample to this. However, in this case there are at least three possible general statements, and the central claim is not couched in language that makes its logical structure clear. Indeed, more acceptable mathematical writing also contains such "suppressed quantifiers" (Selden & Selden, 1995). Thus, being able to re-frame a statement in more appropriate language is closely connected with deciding which objects to check. Emphasizing this when teaching appropriate "mathematical language" may help to explain the need for clarity as well as facilitating checking.

Further research

Having identified these uses of example objects as an important feature of mathematicians' thinking, and as one that is often lacking in students, two questions arise: (1) Does this lack account for the failure of students in introductory proof courses? (2) If so, can teaching that focuses more on the underlying mathematical objects help students to be more successful? The first of these questions is being investigated now in a study comparing explanations of proof attempts by students in the Introduction to Mathematical Reasoning course.

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IMPROVING STUDENT TEACHERS' ATTITUDES TO MATHEMATICS

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*The research results presented in this paper were part of an action research performed with the aims of improving primary school student teachers (**STs**)' understanding of, and attitudes to, mathematics. The teaching strategies used to help STs' improve their understanding and attitudes were similar to the ones suggested for their future use in teaching children. The data indicated that most STs improved their understanding. Some also said that they had improved their liking for the subject and their remarks clearly demonstrated a connection between the affective and cognitive domains. Yet others said that their attitudes towards mathematics had not changed much. The two main aims of this action research remain incompatible in the perception of some of these STs.*

INTRODUCTION

Research has revealed that some primary school teachers and STs demonstrate negative attitudes towards mathematics (e.g., Ball, 1990; Relich and Way, 1994 and Philippou and Christou, 1998). There are many dimensions in the literature about attitudes to mathematics (e.g., Ernest, 1989 and Relich and Way, 1994). The focus of the present study is in the liking dimension of attitudes. Skemp (1989) says that the use of mathematics by adults depends on whether they liked mathematics at school. Considering that primary school teachers have to continue studying and using the mathematics they are supposed to teach, the liking dimension of attitudes was considered more important than the other dimensions.

For Skemp (1976) relational understanding involves knowing both what to do and why it works, while instrumental understanding involves knowing only what to do, the rule, but not the reason why the rule works. Skemp argues that the development of positive attitudes to mathematics is dependent on the type of teaching. Negative attitudes can be generated by a mismatch which occurs when the teacher teaches instrumentally, and the student tries to understand relationally. Baturo and Nason (1996) explains that the main product of instrumental teaching is the lowered self-esteem of students who do not manage to memorise facts and algorithms without meaning. Research shows that some adults with a degree in other subjects (e.g., Quilter and Harper, 1988) and primary school STs and teachers (e.g., Haylock, 1995 and Brown et al., 1997) tend to blame instrumental teaching for their negative attitudes to mathematics. Brown et al. (1990) suggest that an attempt is needed to consider the way by which primary school STs construct mathematical knowledge and what attitudes result from such construction.

THEORETICAL FRAMEWORK AND RELATED LITERATURE

In this study I have adopted a socio-cultural perspective based on the ideas of Saviani (1993) who argues for a pedagogy of liberation that places great emphasis on the acquisition of the cultural content in the school curriculum: "the oppressed does not become liberated if s(he) does not master all that the oppressors master. Therefore, to master what the oppressors master is a condition for liberation" (p. 66). In a similar way, Delpit (1995) points out that teachers should help socially disadvantaged and African-American students "to learn the discourse which would otherwise be used to exclude them from participating in and transforming the mainstream" (p. 165). So STs need not only to develop a more positive attitude to mathematics, but also to acquire a mathematical understanding of an adequate level to face the responsibility of communicating the subject to children and providing effective learning experiences to socially disadvantaged students.

Researchers believe that teachers' attitudes to mathematics can in some way influence their students' attitudes and mathematical learning (e.g., Relich and Way, 1994). Therefore, many teacher educators think that developing positive attitudes toward mathematics should be an important aim in the education of primary school STs and teachers (e.g., Relich and Way, 1994 and Haylock, 1995). STs' attitudes are said to affect: (a) their approach to learning how to teach (Goulding et al., 2002) and (b) the way they will teach in the future (e.g., Ball, 1988) and the classroom ethos (e.g., Ernest, 1989 and Goulding et al., 2002). Teachers are said to rely on memories of themselves as school students to shape their teaching practices (e.g., Ball and McDiarmid, 1990). These memories are also said to affect what they learn from teacher education. Some STs find it difficult to take different approaches from the ones they observed as school students (e.g., Ball, 1988). Ernest (1989) argues that teachers' attitudes to mathematics may influence their enthusiasm and confidence to teaching the subject. This in turn may affect the classroom ethos and consequently affect their students' perceptions of mathematics.

Bromme and Brophy (1986) think that teachers model their attitudes and beliefs during their teaching. In most cases messages are conveyed without teachers' awareness. Yet the most direct influence of primary school teachers' negative attitudes to mathematics on their students' learning appears to be time allocation. Bromme and Brophy point out that "such teachers have been found to allocate more instruction time to subject-matter areas that they enjoy, and less to areas that they dislike" (p. 122). Low time allocation was found to restrict students' opportunities to learn (e.g., Fisher, 1995). Therefore, teachers need to improve their liking for mathematics and to be aware of the benefits of high time allocation especially for activities which have the potential to develop relational understanding.

Most of the attempts to help STs improve their attitudes to mathematics in teacher education seem to involve improving their understanding of the subject. The integration between the re-teaching of mathematics and the teaching of mathematics pedagogy is said to be a way of improving teachers and STs' understanding (Bezuk

and Gawronski, 2003) and attitudes to mathematics (e.g., Weissglass, 1983). Most of the literature reviewed concerning such integration suggests re-teaching mathematics to teachers and STs by using the same methods that could be used to teach mathematics in a relational way to school students. To develop positive attitudes to mathematics in children, primary school teachers must learn how to set up learning experiences that are enjoyable, interesting and give the learner a sense of accomplishment. In order to be able to do this, the teachers must have had such experiences themselves (Weissglass, 1983).

Haylock (1995) and Philippou and Christou (1998) report that improving STs' mathematical understanding has produced positive effects in their attitudes. Haylock (1995) presents several mathematics representations in order to help STs develop understanding of the concepts and procedures in the primary school curriculum. Philippou and Christou (1998) used the history of mathematics in order to help STs understand mathematics concepts. These teacher educators think that improving STs' attitudes is a by-product of the effort to improve their understanding. I took a similar view and in the present study the strategic actions to improve STs' understanding were thought to be helpful in improving their liking for mathematics.

METHODOLOGY

I carried out an action research (Amato, 2001) at University of Brasília through a mathematics teaching course component (**MTCC**) in pre-service teacher education. The component consists of one semester (80 hours) in which both theory related to the teaching of mathematics and strategies for teaching the content in the primary school curriculum must be discussed. There were two main action steps and each had the duration of one semester thus each action step took place with a different cohort of STs. A teaching programme was designed in an attempt to: (a) improve STs' relational understanding of the content they would be expected to teach in the future and (b) improve their liking for mathematics. Four data collection instruments were used to monitor the effects of the strategic actions: (a) diary; (b) pre- and post-questionnaires; (c) middle and end of semester interviews and (d) pre- and post-tests. Much information was produced by these instruments but, because of the limitations of space, only some STs' responses related to changes in their attitudes to mathematics are reported.

In the action steps of the research the re-teaching of mathematical content was integrated with the teaching of pedagogy by asking the STs to perform children's activities which have the potential to develop relational understanding of the subject. The activities were designed with four other more specific aims in mind: (i) promote STs' familiarity with several mathematical representations for each concept (real world contexts, concrete materials, pictures and diagrams, spoken languages and written symbols); (ii) expose STs to several ways of representing and performing operations (with the aid of concrete materials, mentally and with written symbols); (iii) help STs to construct relationships among concepts and operations and (iv) facilitate STs' transition from concrete to symbolic mathematics.

SOME RESULTS

During the action steps of the research, STs' previous attitudes to, and understanding of, mathematics were elicited by two pre-questionnaires. One involving questions related to their liking for mathematics at school and the other asking them what they felt about their understanding of mathematics at school. In both semesters most STs who said they disliked mathematics said they often did not understand mathematics at school. A relationship between liking and understanding mathematics also appeared to exist. The majority of the STs who said they liked mathematics said they often understood mathematics at school. Some of the STs responses to the open questions in the pre- and post-questionnaires and in the interviews revealed further and qualitative evidence about the relationship between the affective and cognitive domains.

The number of first and second semester STs who responded according to a certain theme will be represented by n1 and n2 respectively. The post-questionnaire about understanding was answered by 24 STs in the first semester and by 38 STs in the second semester. Question (1a) of the post-questionnaire about understanding was: "What changes happened in your understanding of the mathematical content discussed in this course component? Give examples". All STs who answered the question said there had been changes in their understanding (n1 = 21 and n2 = 27) and/or in their pedagogical knowledge (n1 = 4 and n2 = 11) of the content discussed in the course component. In the second semester there were five responses about changes in their attitudes towards mathematics and towards certain mathematical content. An example is: "The most meaningful changes were the ones about the rediscovering of mathematics. I learned, for example, that a fraction is not a beast of seven heads". Those responses to a question asking about changes in understanding tends to show that some relationship seems to exist between the affective and cognitive domains for those STs.

The post-questionnaire about attitudes was answered by 30 STs in the first semester and by 40 STs in the second semester. Question (3a) in the post-questionnaire about attitudes was: "Did your involvement with the activities proposed to teach mathematics in the initial grades change, in any way, your feelings about mathematics? Tick your answer." In question (3b) the STs were asked to write about the aspects in the MTCC which they thought had contributed to the changes in their liking for mathematics expressed in the previous close question (3a). In question (3b) some STs (n1 = 11 and n2 = 10) included remarks about changes in their understanding of mathematics. It was interesting to notice the number of those remarks in a question asking them about the aspects in the MTCC which contributed to changes in their liking for mathematics. An example is: "The way to understand and teach fractions was very gratifying for me. I had a lot of difficulty in teaching and mainly in understanding equivalence of fractions".

Some of those remarks were also from STs who said that their liking for mathematics had not changed, like: "Actually I have always liked mathematics, although I had my

difficulties. The interesting thing in this course component was to discover the reasons for the results and to understand the mathematical reasoning". There were also remarks about changes of attitudes in other less focused questions in the questionnaires and interviews. Those responses were considered more valid as they were not prompted by the wording of the question. Some of those responses were also accompanied by remarks about changes in understanding of particular mathematical content. An example is:

> (Interview) Mathematics has always been 'a stone in my shoe'. I always had difficulties in understanding it. To give you an idea, for the first time I am understanding decimals and fractions and the relationship between them. They were never taught to me in that way. I am becoming so happy that even at my age [mature ST probably in the age range of 35 - 40] I decided that I am going to learn more mathematics. There are many things I have learned later in life and mathematics is one of them.

Such responses tended to show that part of STs' dislike for mathematics was related to their instrumental understanding. Therefore, the strategic actions to improve their relational understanding were considered helpful in improving their liking for mathematics. The majority of STs also said that they had liked the idea of using children's activities: "I liked to 'see' the content as a child. The attempt to place yourself in his/her place and to try 'seeing' how (s)he thinks, how (s)he would better understand". Having said all that, it does not mean that there were not problems connected to the idea of attempting to improve STs' liking for mathematics as a by-product of the effort to improve their relational understanding of the subject. For some STs the attempts to achieve affective outcomes were considered incompatible with the attempts to maximise cognitive outcomes.

Many STs in the first semester (n1 = 22) and some in the second semester (n2 = 18) said that their liking for mathematics had increased. The other STs said that their liking continued the same (n1 = 8 and n2 = 22). Although the teaching programme was improved from the first to the second semester, the number of STs who said that their liking for mathematics had increased through the MTCC was smaller in the second semester. This result was influenced by the decision to ask the second semester STs to record some of their practical activities. I was trying to help STs acquire relational understanding at a more reflective and formal level. According to Ball (1990), this "includes the ability to talk about and model concepts and procedures" (p. 458). Recording the practical activities was thought to encourage active learning and STs' reflections on their previous actions with concrete materials. However, a ST made a comparison between the practical activities and their recording which seems to demonstrate how some STs may have contrasted the children' s informal activities with the few teachers' activities included in the programme:

> (Interview) I like the manipulations of concrete materials, but I do not like the reports. I find them boring. [Teacher: Why?] You are dealing with something light that comes spontaneously and then suddenly you have to record these manipulations. It gives you

the impression that we are returning to the traditional way of working.

There were also positive remarks about the reports (n2 = 8). However, there were more negative (n2 = 10) than positive remarks. Yet it was not appropriate to abandon the reports because soon many STs would be teaching primary school children and needed to acquire a strong relational understanding to teach mathematics to the highest level expected of students doing that stage of schooling (Bennett, 1993). Another problem was the number of STs enrolled in each class (42 in the first semester and 44 STs in the second semester). Several STs complained about the class size and explained that the number of STs did not allow me to provide the necessary amount of individual attention. These STs presented moderate to severe difficulties in re-learning the primary school mathematical content in a single semester. They thought that a slower pace and a smaller class would be more appropriate for them.

CONCLUSIONS

The practical activities were time consuming and hard work with large classes, but using children's activities proved to be an appropriate strategy to attempt improving STs' understanding of the mathematics since the majority of STs said, and many indicated in the post-tests, that their understanding had improved. The majority of STs also said that they had enjoyed using children's activities. The use of several mathematical representations, and helping students to construct relationships among concepts and operations, are important strategies in the teaching of mathematics. So the strategic actions and teaching activities did not require any changes in nature; mainly quantitative and timing adjustments were made for the third and subsequent semesters in order to maximise STs' learning during a single semester. More practical and written activities were included for the representations and content that proved to be more difficult for the STs in previous semesters. For this reason certain activities had to be excluded from the programme.

Some STs suggested increasing the teaching time for rational numbers. In the third and subsequent semesters, the activities for rational numbers concepts and operations were started at the beginning of the semester and they continued until the last day of each semester. The number of activities about operations with natural numbers alone was reduced, but there were still many activities about operations with rational numbers which included a natural number part. Through operations with mixed numbers and decimals (e.g., 35¾+26¼ or 24.75-12.53) the STs experienced further activities related to operations with natural numbers and had the opportunity to make important relationships between operations with natural numbers and rational numbers. Yet taking into consideration the difficulties presented by some STs and the time necessary to a practical approach to teaching with big classes, a more appropriate solution would be to offer the MTCC over two semesters with a total of 160 hours as it was suggested by many STs. However, increasing teaching time involves institutional changes. I have been trying to make these changes, but until the time of completion of this paper the problem has not been solved.

Teaching time was the most important constraint affecting STs' learning and attitudes in this study. Changing STs' attitudes proved to be a slow process which required more than one semester of the MTCC. Philippou and Christou (1998)'s intervention involved three course components, but they argue that even more time and challenging experiences are needed to change STs' attitudes that were developed over many years at school. Without deeper understanding of mathematics STs will probably teach mathematics as a set of disconnected rules and algorithms and disseminate even more negative attitudes to the subject among primary school children. One of the most relevant results of the present study was the knowledge I gained about the time needed to help primary school STs acquire a strong understanding of most of the mathematics they will teach.

I could have focussed my teaching on teacher development by adapting content, assessment, principles and aims, but I decided to focus on my social responsibility to primary school children. Ball and McDiarmid (1990) cite the results of two studies that show that curriculum content may be transformed, narrowed or avoided by negotiations made between students and teachers. I could certainly have made my life easier by narrowing or avoiding certain content and the more formal activities in response to STs' complaints. Such negotiations were thought to be socially irresponsible because they would affect STs' learning of mathematics and of pedagogy and this, in turn, could limit their future students' mathematical learning. McDiarmid and Wilson (1991) poses a question connected to this issue and which I think has relevant connections to idea of democracy in schools:

> Waiting for teachers to develop conceptual understandings of the subject matter from teaching it seems both haphazard and callous: Who decides whose children get shortchanged while waiting for teachers to develop understandings of the subject they teach?" (p. 102).

Darling-Hammond (1996) seems to have some sort of answer to this question. Poorly prepared teachers are "assigned disproportionately to schools and classrooms serving the most educationally vulnerable children" (p. 6). According to Darling-Hammond, students' right to learn is directly connected to their teachers' opportunities to learn what is needed to teach well. Without a good preparation, teachers are not able to provide effective learning experiences to socially disadvantaged students.

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TIME AND FLOW AS PARAMETERS IN INTERNATIONAL COMPARISONS: A VIEW FROM AN EIGHTH GRADE ALGEBRA LESSON

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ABSTRACT: This paper compares the way lessons on systems of linear equations unfold in a classroom in the Negev region of Israel with the way they unfold in a Shanghai and Hong Kong classroom. Lessons are viewed as temporal entities describable not only by the nexus of topics they contain but also by how they flow in time. In this light, the lessons in the classroom studied by the authors contrasts strongly with the Shanghai and Hong Kong classroom, the former having a turbulent flow and latter a smooth directed flow. The result is consistent with previous recognized cultural differences in classroom practice and has implications for the bases of international comparisons.

INTRODUCTION

International studies such as the TIMSS have taught us, among other things, that international comparisons are devilishly difficult to make (e.g. Keitel & Kilpatrick, 1999). Even where curricular complexities may be put aside and a common subject agreed upon, Stigler and Hiebert (1999) and others have shown that lesson structure and presentation can vary greatly from country to country, culture to culture. The present paper adduces further evidence for this fact and underlines a crucial aspect of the presentation of mathematical subjects to be taken into account in international studies, namely, the manner in which lessons unfold *in time*.

How mathematics lessons unfold can be described in two complementary ways. One way is according to their logic or rationale. This rationale is determined partly by mathematics itself and partly by teachers' pedagogical styles. But mathematics lessons also unfold with a certain pace; they have a flow, which one may well describe with musical terms such as rhythm and tempo. The logical unfolding of lessons corresponds, roughly, to what has been called *topogenesis*, and the actual flow of lessons to *chronogenesis* (Chevallard, 1985; Brousseau 1999). The different ways time enters into the teaching and learning of mathematics have been studied broadly by Arzarello, Bartolini Bussi, & Robutti (2002). Some of these ways, such as the 'stream of discussion' (Bertolini Bussi, 1992) and Brousseau's 'didactic memory', are examples of 'external time', that is, they are measurable by an observer's clock, while others are examples of 'internal time', which is "primarily individual and unconscious, although its features may be inferred from external clues" (Arzello *et al*., 2002, p. 526). In this paper, we shall be concerned only with external time, though we consider internal time no less important. We shall examine, in particular, how algebra lessons on systems of linear equations flow in time and

how their pedagogical rationale unfolds. The lessons, which are the focus of this paper, were observed in an eighth grade classroom in the city of Beer Sheva in the southern region of Israel. These are compared with lessons in Shanghai and Hong Kong on the same subject matter, as described by Mok, Leung, Lopez-Real, and Marton (Mok *et al*., 2002).

The comparison of our findings with those of Mok, *et al*. showed that while there were some differences between the lessons in Hong Kong and Shanghai, those lessons were far more comparable with one another than they were with the lessons we observed in Beer Sheva. The latter differed strikingly from either Hong Kong or Shanghai. The most prominent divergence of the lessons we studied from those studied by Mok *et al*. was in the way the very idea of a system of linear equations in two unknowns was developed. And it was here that thinking in terms of flow and time proved useful, as we shall see.

RESEARCH SETTING

The research setting for the results to be presented here, both ours and those of Mok, *et al*., is the *Learners' Perspective Study* (LPS), which is an international effort involving nine countries (Clarke, 1998; Amit & Fried, 2002). The project arose out of the video study connected with the Third International Mathematics and Science Study (TIMSS) in which eighth-grade mathematics classes in Japan, Germany and the USA were videotaped and analyzed to identify national norms for teaching practice, norms that might account for achievement scores attained in each country. The LPS expands on the work done in the TIMSS study (which exclusively examined teachers and only one lesson per teacher) by focusing on *student actions* within the context of whole-class mathematics practice and by adopting a methodology whereby student reconstructions and reflections are considered in a substantial number of videotaped mathematics lessons.

As specified in Clark (1998), classroom sessions were videotaped using an integrated system of three video cameras: one viewing the class as a whole, one on the teacher, and one on a "focus group" of two or three students. In general, every lesson over the course of three weeks was videotaped, that is, a period comprising fifteen consecutive lessons. The extended videotaping period allowed every student at one point of another to be a member of a focus group. Needless to say, video technology with its built-in capacity for measuring time proved an invaluable aid in observing how lessons unfold.

The researchers were present in every lesson, took field notes, collected relevant class material, and conducted interviews with each student focus group. Teachers were interviewed once a week. Although a basic set of questions was constructed beforehand, in practice, the interview protocol was kept flexible so that particular classroom events could be pursued. In this respect, our methodology was along the lines of Ginsburg (1997); this methodology was chosen because the overall goal of

LPS is not so much to test hypothesized student practices as it is to discover them in the first place.

The specific case that formed the basis for this paper was a sequence of 15 lessons on systems of linear equations taught by a dedicated and experienced teacher, whom we shall call Danit. Danit teaches in a comprehensive high school. Her 8th grade class is heterogeneous and comprises 38 students, mostly native-born Israelis, but also new immigrants from the former Soviet Union and one new immigrant from Ethiopia.

DANIT'S LESSONS OVER LONG AND SHORT TIME SCALES

Danit developed the idea of a system of equations and its solution over the course of several lessons. These lessons, to the point at which the algebraic solution of systems was first introduced, unfolded as follows:

Lesson 1: Danit went over the notion of a number line, the coordinate system, and the task of plotting individual points.

Lessons 2-3: Equations in two unknowns were introduced; it was highlighted that such equations generally have an infinite number of solutions.

Lesson 4-5: Danit returned to the coordinate system; the students plotted the solutions of linear equations with her guidance, and the observation was made that solutions of such equations indeed lie along lines; the lines were described by Danit as an equation "in a different language." In the course of this discussion, it is important to remark, another representation was subtly brought into play, a table of values.

Lesson 6: The graphic solution of a system of equations was demonstrated—and it was here that Danit first used the phrase 'system of equations'; here too, she considered the meaning of a solution of a system of equations.

Lesson 7: Still concentrating on the graphic solution, Danit showed that there were cases in which the system can have an infinite number of solutions or no solutions.

Lesson 8: The limits of the graphic solution method were discussed, leading the way to purely algebraic solutions to systems of equations.

In this long sequence of lessons, one observes that the lessons shift from graphical representations to algebraic representations, back to graphical representations, back to algebraic representations. *The back and forth movement is not only characteristic of many lessons taken together: in almost fractal fashion, it is also evident within the details of the lessons themselves*. Consider the following segment from lessons 4-5 (these were taught without a break—in itself a point worth noting). In the preceding lesson, the students had discussed equations in two unknowns and had begun to see that they have an infinite number of solutions. Now, in this lesson, Danit makes the transition, which refers directly at the very start to a shift in the form of representation:

> T: [*min. 35*][Writing the equation x+y=6 on the blackboard] Who is willing to tell me what is written here in Hebrew? I want a *translation* into Hebrew, not just "x plus y

equals six"!...You've seen this [*i.e.* an equation like this] in your book, and you know to do with them [referring to the exercises given in the last lesson]—now translate it into Hebrew.

S1: Two unknowns you have to find them

T: Ok, but, a little more…[continues to prod the students]

S2: [*min. 36*] One unknown and another unknown equals six

T: Good, but more, even without the word 'unknown'.

S3: Something and something equals six

…

T: [writes: 'Two numbers whose sum is six'] Find me two numbers whose sum is six. In the language of algebra, we say, 'x plus y equals six'. [*min. 37*] Today, we're going to learn to translate this into another language; we're going to sketch this, that is, what is written here, x+y=6, I don't have write in the language of algebra, I don't have to say it in words: I can sketch it.

For the next ten minutes, approximately, Danit guides the class through a point-wise construction of the graph of line given by x+y=6, including the construction of a table of values. Finally, she observes:

T: …[*min. 47*] What we have obtained in fact is a straight line in the coordinate system that represents this equation. Come, see why. The line stands in place of saying x and y equals 6 …

Although Danit says the line stands *in place* of saying 'x and y equals 6', the equation is still very much present in the ensuing dialogue. Indeed, before moving ahead to the graphic translation, she first moved back to a verbal translation of the equation calling to mind the previous lesson. Moreover, as the dialogue continued other ideas from the previous lesson returned, in particular, that equations in two unknowns characteristically have an infinite number of solutions and that that can be shown by choosing an arbitrary value for x and showing that a value of y can be found.:

T: [The line stands in place of saying x and y equals 6] Now, let's see what happens to a point, any point that I happen to pick on the line. Come, I pick at random this point over here. What is the x of this point? [points at the board]

Ss [several students together]: 7

T: What is y?

Ss: -1

T: 7 plus -1 equals 6 [note: this is what she referred to before as the language of algebra] [*min. 48*] How many solutions are there to this equation?...How many points are there on this line?

S1: 6 [there are, in fact, 6 *marked* points on the line drawn on the board]

S2: 5

S3: 6

S4: Infinitely many

T: Infinitely many!

Thus, in this ten minute segment, the lesson has shifted from an algebraic representation of an equation in two unknowns to a verbal representation to a graphic representation back to an algebraic representation, forward again to the graphical representation, and then back to the algebraic ideas of the previous lessons.

SHANGHAI AND HONG KONG

The Shanghai lesson, as described by Mok, *et al*. (2002) develops the idea of a system of equations in one lesson lasting approximately 42 minutes. It began with a real-life problem concerning the purchase of two kinds of stamps, a constraint was then added, and, by doing so, a system of equations was produced. After 10 minutes of exploratory discussion, the teacher presented a definition in terms of the example. The teacher then lead the class into a purely mathematical context and introduces the idea of a *linear* system in a purely algebraic fashion. Several examples were given to reinforce the definition; the lesson returned to the original word problem, and again in terms of the problem what a solution of a system was defined. Finally the students were given exercises designed to apply the definitions.

In the Hong Kong lesson, which was somewhat shorter than the Shanghai lesson (approximately 35 minutes), the teacher began with a whole class discussion arrangement to review the idea of an equation in one unknown. This discussion was the vehicle for reviewing the notions of 'unknown', 'linear', 'solution'. Having done this, the teacher could then state the topic of the day, namely 'simultaneous equations in two unknowns'. From here, the Hong Kong lesson, like the Shanghai lesson, moved on to a motivating word problem—this time, a problem concerning rabbits and chickens. Again, as in the Shanghai lesson, the definition of a system was given in terms of the word problem. The lesson concluded with a shift to a pure mathematics context in the form of 'worksheet tasks' asking the students to solve systems of equations.

Although Mok, *et al*. (2002) emphasize the differences between the Hong Kong lesson and Shanghai lesson, we were struck by their similarity. They both have a clear structure: a motivating example, central definitions derived from the example, a return to the motivating example (explicitly, in the Shanghai lesson and hinted in the Hong Kong lesson), and exercises that reinforce the definitions. Moreover, this structure is paced to begin and be completed in exactly one lesson.

DISCUSSION

In the Hong Kong and Shanghai classroom, one moved in a very paced manner, in one lesson, from a motivating example to the definition of system of equations and the solution of a system of equations and then to a summation by means of exercises applying the new definitions; in the classroom we observed, one moved in a slow

meandering fashion, over the course of several lessons, through different representations first of equations in two unknowns, then of systems of equations, leading finally to the algebraic solution of a system.

The way Danit's lessons move back and forth between different representations of equations on such a small scale and, simultaneously, on the large scale makes Seeger's (as cited in Arzello *et al*., 2002) comparison of such classroom discussions to *turbulent flow* (as opposed to *laminar flow*) particularly apt. It not only describes the flow of the lesson, it also gives some hint of what the pedagogical effect of such flow is. For in turbulent flow a fluid is constantly being mixed: turbulent lessons are not confused lessons, but ones in which ideas are continually being brought forward and back and compared and contrasted. It is for this reason, we surmise, that Danit, when she finally arrived to the notion of a system of equations, did not see the need to provide an explicit definition: the line representations of the equations and the algebraic equations were continually being mixed, so that the intuitive fact of two lines meeting at a point was immediately being compared to the simultaneous solution of two linear equations.

The turbulent flow of the lessons in the Beer Sheva classroom contrasts strongly with smooth directed flow of both the Shanghai and Hong Kong classrooms. The difference is very likely a cultural one. Stigler and Hiebert (1999) indicated a similar difference between Japanese and American lessons. In Japanese schools, a lesson is considered a perfect whole telling one coherent story. For this reason, a lesson in Japan is not to be disturbed in the middle, and no part of it is to be missed. In American schools, lessons form a series of more or less independent modules: an interruption here or there will not, therefore, ruin the lessons (Stigler & Hiebert, 1999, pp.95-96). Indeed, in our basic set of interview questions, one question asked whether students viewed each mathematics lesson as a single story or as a chapter in an ongoing series, like a 'soap-opera'. In almost every case, the students answered "it is more like a 'soap-opera'."

Accepting the fact of this turbulent flow in the Beer Sheva lessons, one should ask, of course, whether the students benefit from it. Should we, rather, emulate the laminar flow of the Shanghai and Hong Kong lessons? At this point, it is hard to say. We were disturbed to find that when, in the interviews, we asked Danit's students what they understood by 'a system of equations', they had only a vague notion—more than one student identified the system of equation with the coordinate system—even though the same students could often find the solutions to systems without too much trouble. But, to return to musical analogies, it may that in such lessons, such misapprehensions are mere dissonances to be resolved later.

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DISCIPLINED CALCULATORS OR FLEXIBLE PROBLEM SOLVERS?

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Efforts to develop a mathematics curriculum that meets the needs of a modern society are reflected in reform recommendations across the developed world. A common requirement is for students to understand the calculation procedures they are taught and to develop 'number sense'. This paper will analyse students' strategies for calculating in the USA, England and the Netherlands and consider the way these relate to curriculum priorities.

Traditional approaches have emphasised a place value approach to calculations, often modelled on base ten materials, with students taught a standard vertical algorithm. Recent developments emphasise a more thinking approach based on 'number sense'. In the US Standards 'understanding number and operations, developing number sense, and gaining fluency in arithmetic computation form the core of mathematics' in the elementary grades' (NCTM, 2003:1). The National Numeracy Strategy in England (DfEE, 1998) proposes more emphasis on mental strategies with delayed introduction of standard algorithms. Students are expected 'to understand' the four operations and relationships among them and to 'use mental methods if the calculations are suitable' (DfEE, 1999:69). In the Netherlands, the Realistic Mathematics approach emphasises the development of 'models' rooted in concrete situations. Written methods are developed with progressively increasing efficiency using unpartitioned numbers (van den Heuvel Panhuizen, 2001). Implementing change is not straightforward and national proposals are meet with different responses by teachers, educationalists, politicians and the public at large. In the USA, 'math wars' reflect controversies in attempts to change priorities. England and the Netherlands are also subject to different initiatives and although aims are compatible the routes to change involve contrasting practices (Beishuizen and Anghileri, 1998, Anghileri, 2001).

The operation of division

Two distinct procedures for written calculations relate to the *partitive* and *quotitive* models for division (Greer, 1992): repeated subtraction of the divisor (becoming more efficient by judicious choice of 'chunks' that are multiples of the divisor) and sharing, based on a place value partitioning of the number to be divided (used efficiently in the traditional algorithm). The traditional algorithm takes two forms: 'short division' in which the calculation is completed in a single line and 'long division' involving sub-procedures recorded in a vertical format. In England, written division is initially restricted to a single digit divisor with 'informal methods of dividing by a two-digit divisor' (DfEE, 1999). In the Netherlands larger numbers (both divisor and dividend) are introduced to justify the need for a written strategy

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and a standard procedure based on repeated subtraction is taught with efficiency gained through the subtraction of larger 'chunks' (Beishuizen and Anghileri, 1998).. In the USA, the traditional algorithm is introduced for one- and two-digit divisors.

Students' Strategies in England, the Netherlands and the USA

Students written strategies for ten division problems were collected in three countries. English and Dutch cohorts were tested in June of year 5/group 6 when ages were similar (English: mean = 10.21 yrs, s.d. = 0.28; Dutch: mean = 10.32 yrs, s.d. = 0.44). In the USA testing took place later when the mean age of students was 10.75 yrs with s.d. 0.43. These age distributions reflect national policies; in England students' ages determine their class and it is rare to find any variation (Prais 1997). In the Netherlands the age range in many classes will be wider, reflecting a national policy for accelerating able students and repeating years for those who do not reach the required standard. In the US a policy of repeating years also operates.

The study involved students (n=647) from 23 schools in and around small university cities: 10 high achieving English (n=275) schools, 10 Dutch schools implementing curriculum change (n=259) and 3 schools in the USA (n=113). Time constrained the sample to six classes in one state in the USA in one private school, one selective and one non-selective public school. Solutions were collected in individual workbooks using five word problems that varied in their numerical content and their semantic structure, together with five parallel 'bare' problems. The same protocol was used in all classes. Using the students written records, codes for the strategies were established (Anghileri, Beishuizen, & van Putten, 2002, van Putten 2002).

RESULTS

Solutions were predominantly those taught in each country but the frequency of use varied. In England the short division algorithm was used in 53% of all attempted questions, in the US the long division algorithm was used in 81% of all attempts and in the Netherlands the repeated subtraction procedure was used in 60%. The US and English algorithms allowed for no flexibility but the Dutch repeated subtraction method allowed students to choose the number facts to use and could be completed at different levels of efficiency. Other strategies used were generally low-level approaches such as tallying or repeated addition or subtraction.

Single digit divisors

A pair of items involved exact division of a two-digit number (one context and one bare) and another pair involved a four-digit dividend and a remainder:

- q1: 98 flowers are bundled in bunches of 7. How many bunches can be made?
- q6: 96÷6
- q5: 1542 apples are divided among 5 shopkeepers. How many apples will each shopkeeper get? How many apples will be left?
- q10: 1256÷6

Use of the taught algorithms was highest in these questions and percentage use (whether correct or incorrect) is given in Table 1.

algorithm	short division				repeated subtraction				long division			
	q1	q6	q5	q10	q1	q6	q5	q10	q1	q6	q5	q10
England	66	66	70	68								
Netherland					76	74	72	72				
USA									92	91	92	87

Table 1: Percentage use of the different algorithms

The English cohort showed more diversity in the strategies used for these questions with about a quarter of attempts using informal approaches including tallying and chunking using known number facts. The US students used the algorithm almost exclusively. Facilities for these four questions are shown in Table 2.

	q1	q6	q5	q10
England	53	53	35	23
Netherlands	84	81	63	56
USA	96	85	74	69

Table 2: Overall success rates for each question (percentage).

English students did least well with particular difficulty with 4-digit dividends. The Dutch, with their repeated subtraction procedure were more successful. The US success in these four questions reflects extremely high performances in the private school (99% correct overall in these items) and the selective school (91% correct overall in these 4 items) while in the non-selective public school students' success was more modest at 89%, 47%, 47% and 42% respectively for these questions.

Two digit divisors

Two pairs of questions involved two-digit divisors with numbers chosen to encourage informal approaches:

- q2: 64 pencils have to be packed in boxes of 16. How many boxes will be needed?
- q7: 84÷14
- q3: 432 children have to be transported by 15 seater buses. How many buses will be needed?
- q8: 538÷15

Informal strategies were evident in some English and Dutch solutions but rarely in the American students' work. The nationally taught algorithms (English short division; Dutch repeated subtraction algorithm; US long division algorithm) were again most commonly attempted and the following table (Table 3) shows the percentage of items correctly solved compared with the percentage correctly solved using the algorithms in brackets.

	q2	q7	q3	q8
England	48 (8)	55 (14)	23 (15)	13 (8)
Netherlands	75 (50)	76 (40)	53 (45)	53 (43)
USA	73 (55)	81 (68)	36 (35)	64 (64)

Table 3: Facilities (percentage) for each question (success using national algorithm).

Success on these four questions was most limited in the English classes, not least because the short division algorithm is not easily adapted for 2-digit divisors and informal methods were widely used although many students (9%) omitted these problems. In question 3 many US students gave as their answer the result of their calculation and not the number of coaches required.
Due to the nature of the sample, results for the US classes are interesting when comparison is made between the highest and lowest scoring classes (Table 4).

	q2	q7	q3	q8
non selective public school (n=19)	26	11	0 (*10)	11
selective private school (n=19)	100	100	74 (*89)	89

*includes correct calculation but wrong answer

Table 4: Facilities of two US classes for questions involving two-digit divisors

Many wrong answers in the non-selective public school class were due to choice of the wrong operation in the context questions (40%).

Division by ten
Two of the items involved division by ten with a remainder:

- q4: 604 blocks are laid down in rows of 10. How many rows will there be?
- q9: 802÷10

The English students used a variety of strategies with mental methods (an answer given but no working shown) being the most common (34% and 37% respectively). The context question 4 was tackled by 28% using the algorithm while the non-context question was tackled this way by 40%. The Dutch students also used a variety of methods with less difference between the context and non-context questions (54% and 50% use respectively) in the use of the algorithm. A bigger difference occurred with a mental strategy used by 28% and 33% of the students.
As in the other items the US students predominantly used the algorithm in 72% of attempts at question 4 (54% correct) and in 81% of attempts at question 9 (72% correct) and a mental strategy was used by only 8%. None of the US students curtailed the algorithm in any way and full working was shown throughout.

Discussion

It has been proposed that arithmetic instruction is not about designing ways for students to develop facility in calculation, albeit meaningfully, it is about fostering students' underlying arithmetical conceptions (Steffe and Kieren, 1994). Findings of this study suggest this objective is not greatly evident in the written methods for division, and students' approaches in the different countries are starkly contrasted. The US students gained the highest scores overall (72% correct) but since the cohorts from different countries in this study are not directly comparable it is not possible to conclude that this provides the key to successful division computation. Success rates for the non-selective class (Table 4) suggests that the taught algorithm presents considerable difficulties for many students. Facilities are comparable with the English cohort who used the algorithm in 49% of all attempts (Table 5).

	q1	q2	q3	q4	q5	q6	q7	q8	q9	q10
US non selective	89	26	0	26	47	47	11	11	47	42
mean England	76	48	23	46	38	69	54	13	45	22

Table 5: Comparison of facilities (%) in one US class with all English students

The US students are well disciplined with the algorithm used extensively but there was no flexibility (for example no curtailed procedures for division by 10) and there was little evidence of number sense (for example few mental approaches and lack of reference back to a meaningful solution in context). The US students found many bare question, for example division by ten, easier than the context problem, suggesting that they focus on formal calculations more than problem solving.

Where English and Dutch classes can be more readily compared the results show superior mastery with the Dutch approach (see Anghileri et al., 2002). Dutch students predominantly (60%) used 'chunking' with flexibility in the degree of efficiency involved. This algorithm allows individuals to make choices about the number facts they use, thus retaining some ownership of the method rather than replicating a standard procedure. It is suggested that by the time Dutch students are the age of the US students they will have achieved equal or greater success rates although this can only be speculation.

English students used a greater diversity of methods and this fits with the objective of introducing flexibility but appears to be at the expense of competence in calculating. English students used a mental method most (36%) of all three countries for division by ten and were most effective (22% correct) in use of this strategy for these questions. Correct solutions to other questions sometimes (7%) showed inventive strategies for division by a 2-digit divisor which had not been taught, for example for $432 \div 15$ a solution given was $30 \times 15 = 450$ $450 - 15 = 435$ Answer 28r12. Overall English students arrived at correct solutions to 25% of all items using the algorithm but a further 19% using other methods. More diversity in approach can lead to some good strategies but many students are unable to develop their own informal methods

for problems such as 64÷16. At the other extreme it may be questioned whether it is desirable for US students to persist in a rigorous and completed procedure where a less formal strategy would be more efficient and reflect the number sense that is desired. The progressive nature of the Dutch method involves flexibility as it allows students to use the number facts they know without being constrained to the unique steps in the traditional algorithms. With the desire to encourage number sense it is important to question the priority that is given to teaching the traditional algorithm for division, but competence in calculating does not appear to develop where students are left to develop their own methods. A balance needs to be established between flexibility with use of number sense and accuracy in computation. The Dutch approach appears to go furthest in developing an approach that combines both.

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A STATEMENT, THE CONTRAPOSITIVE AND THE INVERSE: INTUITION AND ARGUMENTATION

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The equivalence between a statement and its contrapositive is so obvious for an expert that, usually, he does not need any explanation. In this paper, we shall examine the argumentations which students produce in order to justify a statement that, in their opinion, is equivalent to a given statement. We shall observe that the most common argumentations come out from the effort to adjust the proof of the first statement to the second one. Analysing these argumentations, it will emerge that for the students the (false) equivalence between a statement and its inverse is intuitive and the (true) equivalence between a statement and its contrapositive is not intuitive.

INTRODUCTION

In this paper, we investigate the equivalence between a statement and its contrapositive from cognitive and didactical points of view. Given a statement $p \rightarrow q$, the contrapositive is the statement $\neg q \rightarrow \neg p$, the inverse is $\neg p \rightarrow \neg q$ and the converse is $q \rightarrow p$. A statement and the contrapositive are equivalent, then, if we have proved the statement, the contrapositive is proved too. A statement and the inverse are not equivalent; it happens that a statement is true but the inverse is false; in the same way a statement and the converse are not equivalent.

Usually, in school practice, teachers and text-books do not pay much attention to the equivalence between $p \rightarrow q$ e $\neg q \rightarrow \neg p$, probably because it is considered to be obvious: it is common opinion that this equivalence is a natural way of thinking that can be spontaneously used in mathematics. For example, in a discussion about proof from didactical point of view, an university teacher stated that "*the proof by contradiction is based on the fact that* $p \rightarrow q$ *and* $\neg q \rightarrow \neg p$ *say the same thing*". It was so obvious for him that he did not need any explanation. Moreover, the link between the two statements was so strong for him that he did not seem to be talking about a logic equivalence: the two statements "*say the same thing*", as they were two different forms of the same statement.

On the other side, some researches in education dealt with this equivalence and showed that it seems to be content dependent and, in some cases, to be not so obvious. For a rich bibliography, see Fischbein (1987, pp. 72-81).

Moreover, some researches in mathematics education have pointed out some epistemological, cognitive and didactical obstacles for students when they have to understand or to produce proofs by contradiction, that is proofs whose validity is based on the equivalence between a statement and the contrapositive. See, for example, Wu Yu et al. (2003), Antonini (2003a, 2003b, 2001), Bernardi (2002), Reid

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(1998), Epp (1998), Thompson (1996), Barbin (1988), Leron (1985), Freudenthal (1973).

The aim of this work is to investigate the argumentations proposed by secondary school students (grade 10) when they justify the validity of the contrapositive, the converse and the inverse of a given statement. In the theoretical framework on intuitions exposed by Fischbein (1987, 1982), analysing the argumentations, we can examine and discuss the intuitiveness of these equivalences.

THEORETICAL FRAMEWORK

This study is based on the theoretical framework on intuitions exposed by Fischbein (1987, 1982). In this work, an intuition is "*a representation, an explanation or an interpretation directly accepted by us as something natural, self-evident, intrinsically meaningful, like a simple, given fact"* (Fischbein, 1982, p.10).

Moreover, it is very interesting the following classification:

> *"We have, then, three different kinds of convictions. One is the formal extrinsic type of conviction indirectly imposed by a formal (sometimes a purely symbolical) argumentation. The second is the empirical inductive form of conviction derived from a multitude of practical findings which support the respective conclusion. The third is the intuitive intrinsic type of conviction, directly imposed by the structure of the situation itself" (Fischbein, 1982, p.11).*

If one knows something intuitively, "*he will not feel the need to add something which could complete or clarify the notion (for instance an explanation, a definition, etc.)"* (Fischbein, 1982, p.10). For example, what the university teacher said about proof by contradiction (see above) is, for him, "*something natural, self-evident, intrinsically meaningful, like a simple, given fact"*, i.e. an intuition.

From the mathematical point of view, the proofs of the equivalence between a statement and the contrapositive are based on the principle of excluded middle and they can be formalized in the logic of proposition, for example through the truth tables. We underline that these are meta-theoretical proofs: the object of these proofs is the validity of a statement in relation to the validity of another statement.

Then, it will very interesting to observe the explanations that students propose to support or to reject the equivalence between a statement, the contrapositive or the inverse or the converse.

We expect the students not to produce meta-theoretical argumentations. We rather expect them to propose specific argumentations that depend on the given statement. Through the analysis of these argumentations we shall investigate the intuitiveness of the equivalence between a statement and the contrapositive and the non equivalence between a statement, the inverse and the converse.

METHODOLOGY

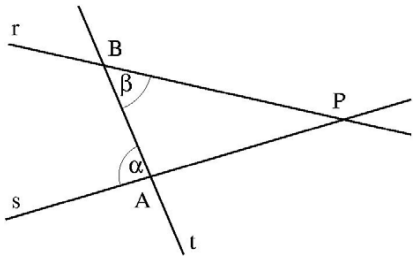
This work is a part of a wider study about proof by contradiction exposed in Antonini (2003a) and then is based on the methodology of that research. In particular, for the research explained here, we observed 46 secondary school students (grade 10) through discussions, reports written by the students about the discussions, a test, and a report in which students had to justify their answers in the test. In this paper, we describe and analyse what emerged from the discussions and from the reports about two particular mathematical problems. To make the exposition clear, we discuss the results according to the mathematical problem they refer to.

THE PROBLEM OF THE PARALLEL LINES

The episode is part of the regular didactical activity in two classes. The students have just proved the following statement (that we call the *main statement*):

Main statement: *Let r and s be two intersecting lines and let t be a transversal. Then the alternate interior angles are different.*

The proof proposed by the students is based on the theorem of the exterior angle: in a triangle an exterior angle is bigger than every non adjacent angle. In this case, in the triangle ABP, we can say that $\alpha>\beta$, and then $\alpha\neq\beta$.



After the proof, the teacher asked the students: *can you formulate a statement which contains parallel lines in the hypothesis or in the thesis, and which does not need to be proved because we already know that it is true from the validity of the main statement?*

During the discussion, students proposed two statements, the inverse and the contrapositive:

inverse: *If r is parallel to s then the two alternate interior angles are equal*

contrapositive: *If the alternate interior angles are equal, then r is parallel to s.*

At home, the students wrote a report in which they exposed their opinions. It emerged that 20% of the students chose the contrapositive, 20% of the students chose the inverse, 50% of the students chose both the statements and 10% of the students completely misunderstood the task or the statement.

The most common argumentations to justify the statement come out from the effort to adjust the proof of the main statement to the inverse or to the contrapositive. The students, instead of producing meta-theoretical argumentations, proposed argumentations based on the same theoretical reference (that is the exterior angle

theorem) of the proof of the main statement. By this kind of argumentation students could support the validity of both the inverse and the contrapositive: these argumentations do not allow to identify the statement equivalent to the given one. For example, Fabiana, after the proof of the main statement, writes:

> "In this way, using the exterior angle theorem, we can prove and be sure that two intersecting lines and a transversal make a triangle and the two alternate interior angles are different.
>
> Probably, to prove that two parallel lines and a transversal [make] two equal alternate interior angles, we have to think in the same way, using the ideas and the theorems previously proved."

Fabiana thinks that, to prove the inverse, "*we have to think in the same way*". Then, she explains what she means in the following way:

> "We can prove the exterior angle theorem, that is that $\alpha>\beta$ and then that $\alpha\neq\beta$ just when we have a triangle formed by the intersecting lines. Then, when the lines are parallel, they cannot make a triangle because they do not intersect each other and then we cannot apply the theorem and then we cannot prove that $\alpha>\beta$ and then, if the two lines are parallel, $\alpha=\beta$."

In the same way, Fabiana supports the contrapositive using the exterior angle theorem:

> "If I have in the hypothesis that $\alpha=\beta$ and in the thesis that r//s I can reverse the argument because I know that $\alpha=\beta$ and then α is not bigger than β then I can deduce that the lines do not make a triangle and then I can not apply the exterior angle theorem."

Explanations like this are very common. We can identify two elements in these argumentations: one about the content, one about the structure. Firstly, they are based on the same argument of the proof of the main statement, that is the exterior angle theorem. Secondly, the structure of these argumentations is always the following: *if we can apply a particular theorem, we deduce a certain conclusion, otherwise we deduce the opposite of that conclusion*.

Other students answered in a very different way: they proposed the inverse of the main statement and they did not need to give any argumentation for this choice. The following episodes are just two examples of many similar protocols:

158. Achille: if [the lines] are parallel, the angles are equal.

159. Teacher: You are thinking about the first statement [the inverse]. How are you going to prove it?

160. Achille: because… if they intersect each other … is so… if they do not intersect … it is not so…

161. Teacher: Is this a proof?

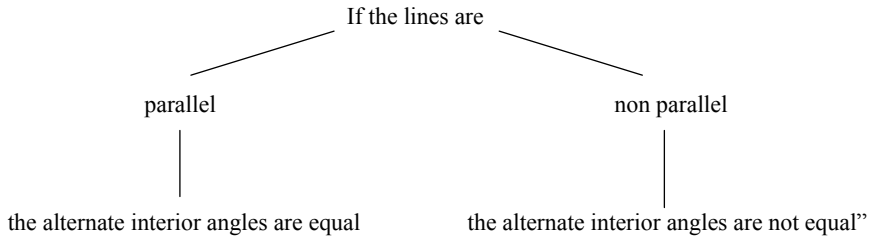
162. Achille: No.

163. Teacher: But it is a convincing argument for you.

164. Achille: Eh… if we begin from two intersecting lines… the angles are different… if we begin from non intersecting lines, the angles will be equal!

Achille is convinced, it is so evident for him that he does not need to propose any further argumentation. Also Gianna, in the written report, answers in a similar way:

"In my opinion, it is like a schema:



According to Fischbein (1987, pp. 72-81), we can conclude that for Achille and Gianna the equivalence between a statement and the inverse is an intuition. In this regard, the Franca protocol is very enlightening. She proposes a logic argumentation based on the truth tables to justify the equivalence between a statement and the contrapositive[1]: her explanation is the only one we can consider a proof. Nevertheless, she also asserts the equivalence between the statement and the inverse:

> "But, in my opinion, it is correct to say also that, if r//s then we do not have any elements to prove that $\alpha \neq \beta$ and then we have to accept the thesis $\alpha = \beta$."

Even if Franca writes a meta-theoretical proof to support the equivalence between a statement and the contrapositive, she proposes an argumentation based on an intuition which can not be formalized in logic: "*we do not have any elements to prove that $\alpha \neq \beta$ and then we have to accept the thesis $\alpha = \beta$*". We explain these phenomena describing two *convictions* (Fischbein, 1982) that can coexist: the first one is the intuitive *conviction* that a statement and the inverse are equivalent; the second one is the non intuitive *conviction* that a statement and the contrapositive are equivalent. Other students, like Franca, write that the contrapositive is the equivalent statement, but the more intuitive *conviction* emerges and they affirm the equivalence of the inverse too. With words of Fischbein, the first one is an *intuitive intrinsic type of conviction*, the second one is a *formal extrinsic type of conviction*. We remark that the last one does not seem to have any effects on the first *conviction* which can still continue to be an obstacle.

[1] We remark that the use of the truth tables was never used for problems of this kind neither by these students nor by the teacher.

THE PROBLEM OF THE QUADRILATERAL

After the episode of the parallel lines, the teacher spent some lessons to justify the equivalence between a statement and the contrapositive and the non equivalence between a statement and the inverse. The teacher's explanations were based on two different contents: the proof by the truth tables and the familiar examples expressed in natural language. Three weeks later, students were asked to answer a test with two questions:

1) Write down the hypothesis and the thesis of the statement: *If the angles of a quadrilateral are equals, the diagonals are equal.*
2) Let us suppose that the statement of question 1) is proved. Which of the following statements is consequently proved?
    a) If the angles of a quadrilateral are not equal, the diagonals are different.
    b) If the diagonals of a quadrilateral are different, the angles are not equal.
    c) If the diagonal of a quadrilateral are equal, the angles are equal.

Question 1) is useful for us to understand if the answers to question 2) could be dependent on an incorrect identification of the hypothesis or of the thesis. Every one of the 43 students answered correctly to the first question, then we have to look for other elements to explain the students' difficulties with the second question. Calling "*main statement*" the one about the quadrilateral in question 1), the statement in a) is the inverse, the statement in b) is the contrapositive and the statement in c) is the converse. The data of the answers to the second question are in the table:

inverse	contrap.	converse	inverse and contrap.	inverse and converse	contrap. and converse	Inverse, contrap. and converse
15	9	8	6	1	2	2
35%	21%	19%	14%	11%		

We underline that, in this case, the inverse is false; in the problem of the parallel lines, the inverse is true even if it can not be proved from the main statement. Nevertheless, a lot of students chose the inverse as their answer, both in the problem of the parallel lines and in the problem of the quadrilateral.

After the test, the teacher asked the students to write down a report explaining the reasons for their answers.

In this case, the students did not have the proof of the main statement, then they could not use this proof to justify their choices like they did in the problem of the parallel lines. Then they produced a lot of examples, in order to recognize and to argument

the correct statement. We remark that, in this way, the students looked for the valid statement among the three proposed: they did not refer to the equivalence between the chosen statement and the main one. Just one student used correctly the truth tables to answer the question. Nevertheless, Marilena, after having proved that the main statement was not equivalent to the inverse using the truth tables, wrote that "*we can not deduce [the inverse] from the proved theorem, even if it is intuitively true*". The presence of the intuitive *conviction* about the inverse is evident but now Marilena checks her intuition through a formal tool. However, the fact that a statement and the inverse are not equivalent is a *formal extrinsic type of conviction*, but it is not yet an intuition.

CONCLUSIONS

We have analysed the argumentations that students proposed to justify the equivalence between a statement and the contrapositive or the inverse: it emerged that most of these argumentations come from the proof of the main statement. When the students did not know this proof they have justified such equivalence producing a set of examples. In every case, their aim was to look for a valid statement and not a statement equivalent to the given one.

According to Fischbein (1987, pp. 72-81), we can affirm that the (false) equivalence between a statement and the inverse is an intuition while the (true) equivalence between a statement and the contrapositive is not an intuition. Moreover, the argumentations proposed by the teacher, based on the truth tables and on familiar examples, did not help the students: just few students used these type of explanations to justify their answers, and even for them the topic did not become intuitive.

According to Antonini (2003a, 2003b), Thompson (1996) and Freudenthal (1973), it is fruitful to set up situations in which students spontaneously produce *indirect argumentations*, that is argumentations like "*...if it were not so, it would happen that...*" (Antonini, 2003b). Moreover, it is important to guide the students to the awareness of the structure of their argumentations, so that the knowledge of the equivalence between a statement and the contrapositive and of the non equivalence between the statement and the inverse becomes an intuitive knowledge. As Fischbein wrote:

> *"The training of logical capacities is a basic condition for success in mathematics and science education. We refer not only to a formal-algorithmic training. The main concern has to be the conversion of these mental schemas into intuitive efficient tools, that is to say in mechanisms organically incorporated in the mental-behavioral abilities of the individual" (Fischbein, 1987, p.81).*

It is important that the contrapositive becomes a way of thinking, an intuitive knowledge, in order to assume the features of a thinking tool.

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SOLUTION - WHAT DOES IT MEAN? HELPING LINEAR ALGEBRA STUDENTS DEVELOP THE CONCEPT WHILE IMPROVING RESEARCH TOOLS

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Twelve linear algebra students were interviewed about the concept of a Solution of a System of Equations. The interviews were analyzed using APOS tools, in particular the ideas of Action, Process, Object and Schema, and Genetic Decomposition. The analysis of the interviews revealed several misconceptions of Solution. The analysis also revealed shortcomings of the questionnaire that was used in the interviews: It did not permit making a distinction between lack of knowledge and partial knowledge. Research tools were improved (questionnaire, GD, and suggestions for teaching materials) and prepared for the next cycle of research.

THEORETIC PERSPECTIVE

The research reported here is part of a broader research effort conducted by the RUMEC (Research in Undergraduate Mathematics Education) group, which is dedicated to research within the scope of the theory named APOS. APOS is an acronym for the ideas of Action, Process, Object and Schema. This theory is an elaboration of Piaget's cognitive theory (Piaget, 1975) for learning mathematics. Detailed description of this theory can be found, for example, in Asiala et al., (1996). Here we will only describe such elements of this theory that are used in this report.

Action

According to APOS, the development of every concept begins in the learner's mind with an action. At this level the learner can only perform the action one step at a time. For example, given a system of linear equations with *n* unknowns, as well as several tuples and matrices of different sizes, students are asked which of the givens is a possible solution. If the students start substituting each tuple separately, we suspect that they cannot imagine in advance whether a given tuple can be substituted and hence be a prospective solution. The theory accounts for such inability by the explanation that at the action level, the learners are able to complete the action step after step, but cannot think of it as a whole and predict its outcome; Sometimes they can also not describe it verbally.

On the other hand, the behavior described above might indicate that *substitution in order to check equality* is the action used by these students as the starting point for

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conceptualizing *solution*. We will show findings indicating the possibility that some of our students used a different action for the development of the concept *solution.*

Process

When a learner successfully predicts the outcome, invents shortcuts and can describe the action verbally without actually performing it, we say that the concept has developed in his or her mind to the level of a process. For example, in the situation described above, if students can point to appropriate tuples as possible candidates for solutions, as well as explain the relation between the tuples' length and the number of unknowns - we might infer that their conception of *solution* is at least at the process level. They are able to envision the action of substitution without actually performing it.

Object

When the learner can already perform a new mathematical operation on the process itself, or consider the process as an element of a set of processes of its type, with rules or operations within that set – we say that the concept has developed in the learner's mind into an Object. Examples:

a) If students, when confronted with a system of linear equations are asked "what does a solution look like", are able to describe the form of *a solution* of that system - we may conclude that their understanding of *solution* is at the object level.

b) Another example of object level of *solution* concerns understanding the rule that the sum of two solutions of a homogeneous system is also a solution. Such understanding also requires an object understanding of solution, for otherwise the student would not be able to relate to the binary operation "sum of two solutions".

Genetic Decomposition

Learners can begin the development of a certain concept out of different actions. These will result in different understandings of the concept (as opposed to different levels of understanding). A genetic decomposition of a specific concept consists of a detailed description of such possible action, and the typical mathematical behaviors and reactions of a student who has developed that same concept, beginning with that action, throughout the different levels (Action, Process, Object and Schema). Hence, a satisfactory GD can first of all be used as a diagnostic tool, providing the teacher and investigator with insight into the learner's situation in the development of the concept. In addition, it helps the teacher and material developer to provide the student with activities which will enhance his progress in developing his understanding of the concept through the different levels Action-Process-Object-Schema.

It should be emphasized that a GD of a concept is in itself a developing structure. Also, it cannot be assumed to be unique. (DeVries et al., 2001.)

METHODOLOGY OF THE REPORTED RESEARCH

The research reported here is the first cycle in such a research program. Fifteen students at a Teachers' College took a one-semester course in linear algebra. Of the 15 students, 12 were interviewed shortly after completing the course (the rest did not show up for the interview). The interviews were conducted individually. They consisted of a structured questionnaire, which each student solved in front of the interviewer and discussed his work with her. Each interview lasted about 45 minutes, and was video-recorded. Question 1 was constructed to investigate the concept of solution (see appendix 1).

The purpose of this part of the interview was two-fold: Getting to know students' ideas about *solution*, and getting started with a first version of a GD for this concept.

We were not interested in statistical data about the occurrence of the different reactions, as the group of interviewees was not sampled and hence not representative.

FINDINGS

What we discovered after interviewing the students was that our questionnaire was not adequate for providing sufficient insight into our research questions. The responses to this questionnaire gave us little information about the constructions that students have made in their understanding of the concept *solution of an equation*. This basically occurred because only students who had constructed a solution of an equation as an Object could answer the questions in a meaningful way. Also, this provided little possibility of distinguishing between "no understanding" and "partial understanding" (between the Action or Process levels).

In the first part of this report we describe some of the responses obtained. Their analysis leads to suggesting an improved protocol for the interview, an initial version of a GD for *solution*, and a proposed teaching sequence.

Response type 1: *What does it mean "What does a solution look like"*

Some students at first tended not to reply to the question *What does a solution look like?*. Some explained they did not understand the question:

Interviewer: We now deal with question 1:A.
Hersch: What does it mean "What does a solution look like"
Interviewer: What do you think it will look like?
Hersch: The solution here is a number?
Interviewer: What number for instance? Can you give an example?
Hersch: No, because I don't understand the question.

Upon further probing Hersch concluded:

> "If it's a solution of such a thing, there are four elements here. ...So we should also get four solutions for such an equation."

Hersch really could only think of a solution as a number. Consequently, questions regarding sums of solutions and number of solutions did not provide insight into Hersch‘s thinking.

Response type 2: Memorized rules about the sum of two solutions.

Lin relied on a memorized rule rather than reason in answering questions about solutions.

Interviewer: Okay suppose we had two solutions u and v. Is u+v also a solution?
Lin: Yes by the rule that the sum of two solutions is also a solution.

The interviewer questioned him as to why this rule was true and all Lin could do was repeat the rule.

There is another indication of the fact that the rule students quoted was memorized without understanding. Some of them applied it both to homogeneous and to non-homogeneous systems. Example:

Earlier in the interview, Tania was certain that the sum of two solutions of a homogeneous system is also a solution. Now she is asked about a non-homogeneous system:

Interviewer: Here is an equation. Is it homogeneous?
Tania: No.
Interviewer: Why not?
Tania: It is not equal to zero.
Interviewer: If we took two solutions of these infinitely many, and added them the way one adds vectors, will it also be a solution?… We’ll take an actual sum. Will the sum also be a solution?
Tania: Yes.

We suspect that students who provided responses of types 1 and 2 have certainly not developed *solution* into the Object level. The deficiencies of our questionnaire prevented us from tracing any lower levels of knowledge, if such existed.

Response type 3

Some students confused a solution of an equation (or system), with the constant "Right Hand Wing” of the equation (or system). This might be related to findings about the concepts associated by college (as well as k-12) students with the equality sign. Research shows that students of different ages tend to interpret the equality sign to mean: ”the result is”, rather than symbolizing equivalence (such as the equivalence accomplished when substituting a solution into both sides of the equations). See for example Kieran, 1981.

Response type 4: *Solution* as *solving*

Several students in response to the question of what a solution looked like, proceeded to solve the equation. Tania provides an example of this. She correctly described the procedure for finding the solutions of the equation. She did not think

of substitution to verify equality being a defining property of being a solution. It was apparent that most such students were confusing the concepts of *solution* and *solving*.

Our explanation of responses of type 4 is that for these students the concept of *solution* developed out of the action of solving the equation (or system of equations), rather than the action of substitution. By this we mean the solving methods (algorithms) they used (such as Gaussian elimination, or any other). Such algorithms are difficult to interiorize, and do not make it easy for the student to predict the form of the outcome, the solution, without actually calculating it. Using APOS terms, we suspect that for these students, *solution* as *solving* is at the Action level of development, and we know that when a concept is still at that level of its development, the student can only perform the action one step at a time. Hence their tendency to start solving when asked about the solutions. Another characteristic of this level is that the student has no ability to predict the outcome without actually performing the action. Here - the students could not predict the mathematical form of the solution, of the outcome of the solving procedure, before they actually carried it out. We predict that using the action *substitution* as basis for the development of the concept *solution* will end up with easier interiorization of the action and its transforming into process.

AN INITIAL GENETIC DECOMPOSITION FOR *SOLUTION*

We will sum up this discussion with a proposal of an initial GD for a *solution of an equation*:

An equation is an ordered pair of functions (*f*, *g*) with a common domain and a common co-domain. A *solution* of an equation is an element s of the domain for which f(s)=g(s). The *solution set* of an equation is the set of all solutions.

Note: In linear algebra we are usually interested in linear functions from F^n to F^m , and the function g is constant. The pair (f, g) can be represented by means of a system of linear equations, matrices or linear transformations.

SUGGESTED LEARNING SEQUENCE

Action level of the concept *equation*, including *solution*.

We propose to start by helping students construct the Action level of the concept of *equation*, including the ability to identify the two functions, their common domain and co-domain, and *solution* in the sense of an element of the domain, the substitution of which produces a true equality. Here we propose to have them substitute elements of the domain into the two functions and learn to identify solutions and non-solutions.

Process level of *equation* (including *solution*)

Students should be taught to identify the functions and their domains and co-domains for various forms of equation, without being given examples of elements

for substitution. They might also be asked to describe the format of possible solutions and non-solutions.

Object level of *solution*

Here we recommend working on finite fields. Students might be asked to design a computer program that receives an equation as its input and produces the solution set of the equation as its output. The program does this by substituting and checking all the elements of the finite $\mathbf{Z}_p^n$ for equality. The programming language ISETL was found to be adequate for that purpose (see Asiala et al., 1996). Later, when we give students a system of equations over an infinite field they will face a need for other methods, as the previous method has now become useless for both computers and humans. Learning to solve algorithms will now include the understanding of what the algorithm does: It produces only substitutions that are truth-valued, and all such substitutions.

IMPROVED QUESTIONNAIRE

In appendix 2 we presented our improved questionnaire. In the first interview (Appendix 1) most of the questions required an object level understanding of *solution* in order to give any answer at all to the questions. Consequently, we did not get any sense of the level of cognitive development regarding the concept. So in the second round we tried to probe more fundamental constructions regarding the *solution*. For example, in Question 1 we give the student a specific ordered pair and ask if it is a solution, rather than asking what *a solution* would look like. This would indicate at least an Action conception if the student substitutes into the equation.

Further on, in Question 2, checking by substitution whether a matrix is a solution demands some tedious calculations. If the students have reached process conception of solution, they might realize without actually substituting, that the 3x2 matrix (b) is non-substitutionable. Thus we can identify a process conception of solution.

CONCLUSION

In the present research cycle we learned a little about what students think of *solution.* We also recognized the deficiencies of our research tools. As a result, we constructed an improved questionnaire, an initial version of GD for *solution*, and a suggestion for a teaching sequence resulting from that GD. We are now ready for the next cycle of our research.

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Appendix 1: The questionnaire of the reported research

A. What is a solution of this equation (what does it look like)? $\mathbf{3x_1+2x_2-x_3+x_4=5}$

How many solutions does the equation have?

Is the sum of two solutions also a solution? What about a scalar multiplication?

B. What does a solution of this equation look like? $x_1\cdot\begin{pmatrix}1\\1\\0\end{pmatrix}+x_2\cdot\begin{pmatrix}1\\2\\2\end{pmatrix}+x_3\cdot\begin{pmatrix}1\\0\\1\end{pmatrix}+x_4\cdot\begin{pmatrix}1\\1\\1\end{pmatrix}=\begin{pmatrix}3\\1\\0\end{pmatrix}$

Which of the following might be a solution? **a**. $\begin{pmatrix}1&0\\2&1\end{pmatrix}$ **b**. **7** **c**. **(1, 0, 1, 7)** **d**. $\begin{pmatrix}2\\0\\1.5\\7\end{pmatrix}$

How would you check whether it is a solution?

C. Here is a homogenous system of equations Ax=0. Suppose each of the vectors u and v is a solution of this system. What do you think of the vector u+v? Is it a solution of the system or not?

(If no answer) Would you like to use an example?

(If no answer) Would you like me to present an example to you?

Here is an example: $\begin{pmatrix}1&2&3&0\\0&1&1&2\\1&0&1&1\\-2&-3&-5&-3\end{pmatrix}\cdot\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix}$ What would be a solution of this system?

Which of the following vectors is a solution $\begin{pmatrix}1\\2\\0\\0\end{pmatrix}$, $\begin{pmatrix}-2\\-2\\2\\0\end{pmatrix}$, $\begin{pmatrix}8\\8\\-8\\0\end{pmatrix}$? How can we check?

Is the sum of two vectors which are solutions, also a solution?

D. What about a non-homogenous system? How does it differ from a homogenous system?

Here is a non-homogenous system: $\begin{pmatrix}1&2&3&0\\0&1&1&2\\1&0&1&1\\-2&-3&-5&-3\end{pmatrix}\cdot\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}=\begin{pmatrix}1\\1\\0\\1\end{pmatrix}$

Suppose each of the vectors u and v is a solution of this system. What do you think of the vector u+v? Is it a solution of the system as well? How can we check/prove?

E. A and B are nxn matrices of the same order. What would be a solution of such an equation: AX=B

Appendix 2: The new questionnaire

1. Consider the equation $2x_1 + 3x_2 = 6$

(a) Explain why **[6,-2]** is a solution. (b) Find another solution.

(c) What is the sum of the solution in (a) and your solution in (b)?

(d) Is the sum you found in (c) also a solution? Why or why not?

(e) Is a scalar multiple of a solution also a solution? Why or why not?

(f) How many solutions does this equation have? Explain.

2. Consider the equation: $\begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 2 \end{bmatrix} X = \begin{bmatrix} 7 & 0 \\ 3 & 0 \\ 9 & 0 \end{bmatrix}$

(a) Is $\begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix}$ a solution? Why or why not? (b) Is $\begin{bmatrix} 1 & 0 \\ 4 & 1 \\ 6 & 2 \end{bmatrix}$ a solution? Why or why not?

3. Consider the system of equations: $3x_1 + 2x_2 - x_3 = 0 \quad x_1 - x_2 + x_3 = 0$

(a) Is [2,-3,0] a solution? Why or why not? (b) Is [3,-2,-5] a solution? Why or why not?

(c) Does the system have more than one solution? Explain.

(d) Find the solution set of the system.

(e) Is the sum of two solutions also a solution? Why or why not?

(f) Is a scalar multiple of a solution also a solution? Why or why not?

4. Consider the equation: $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 2 \end{bmatrix} X = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 2 \end{bmatrix}$

(a) Is $\begin{bmatrix} 0 \\ 1 \\ 0 \\ 2 \end{bmatrix}$ a solution to this equation? Why of why not? (b) Is $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ a solution to this equation? Why or why not?

ORGANIZING WITH A FOCUS ON DEFINING
A PHENOMENOGRAPHIC APPROACH

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This paper is based on the preparatory study of a doctoral study in which we learned to consider defining in the realm of organizing. In particular, having engaged students in a situation based on "equivalence relations" (from an expert point of view), we report two different ways of organizing the given situation. One of them results in a "new" definition of equivalence relations, and consequently a new representation for them, that seems to be overlooked by the experts.

INTRODUCTION

Definitions are inextricable parts of higher courses in mathematics. They give definiteness to the concepts to be taught in the course; they designate whether something is an example or not, and they are used in proofs. Embracing those referential and inferential aspects, definitions are tools to organize the content of the course; or in general, as Freudenthal (1983, pp.ix-28) says, they "have been invented as tools to organize the phenomena...phenomena from the concrete world as well as from mathematics...". In addition, incidental to their role as means of organizing, they embody two kinds of the lecturers' (or the mathematicians') choices, first, their choices of what appears to be important to be defined, and second, their choices between possible definitions of what is defined.

Accordingly, researchers examine possible ways of introducing definitions when developing new concepts. According to Freudenthal (ibid, p.32), one possibility is describing definitions in their relation to the situations of which they are the means of organizing; then, "starting from those phenomena that beg to be organized and from that starting point teaching the learner to manipulate these means of organizing."

This study has partly adopted Freudenthal's plan in that students were engaged in a situation that begs to be organized, though it aims at investigating the ways that students organize a given situation, rather than teaching them any particular ways of organizing that. In particular, this study is a phenomenographic investigation of what counts as defining when students organize a situation in which they have an opportunity to experience referential and inferential aspects of definitions in conjunction with their choices of the ways of organizing the given situation.

Adhering to a phenomenographic research approach, the study was conducted by holding individual in-depth task-based interviews, in which a task was used for querying students' referred concepts. All the interviews were audio-taped, and they were analyzed to explore *what students used to organize the situation, and what they did to organize the situation.*

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The importance of the issue can be clearly seen in other researches regarding defining. For example, Mariotti and Fischenbein (1997), in a teaching experiment, brought defining into the realm of students' experience. In their experiment, amongst others, two phases are worth considering, first, introducing a problem situation in which "the concept to be defined functionally emerges from the solution of a problem", and second, the indispensable and involved role of the teacher in their experiment, to guide students to overcome the conflict between "…the spontaneous process of conceptualization and the theoretical approach to definitions". Nevertheless, they repeatedly report the students' unforeseen difficulties to transcend the concrete situation to reach to the intended "systematic organization of concepts".

This and our initial data have led to the idea of looking not for how students define the intended concepts, but which concepts they determine are important for organizing the situation and what part defining play in that organization.

PREPARATORY STUDY

This paper is based on our preparatory study in which a smallish sample of students was engaged in the following tasks (see Table 1&2):

A country has ten cities. A mad dictator of the country has decided that he wants to introduce a strict law about visiting other people. He calls this 'the visiting law'. A visiting-city of the city, which you are in, is: A city where you are allowed to visit other people. A visiting law must obey two conditions to satisfy the mad dictator: 1. When you are in a particular city, you are allowed to visit other people in that city. 2. For each pair of cities, either their visiting-cities are identical or they mustn't have any visiting-cities in common. The dictator asks different officials to come up with valid visiting laws, which obey both of these rules. In order to allow the dictator to compare the different laws, the officials are asked to represent their laws on a grid such as the one below. (See the results section)

Table 1

The mad dictator decides that the officials are using too much ink in drawing up these laws. He decrees that, on each grid, the officials must give the least amount of information possible so that the dictator (who is an intelligent person and who knows the two rules) could deduce the whole of the official's visiting law. Looking at each of the examples you have created, what is the least amount of information you need to give to enable the dictator deduce the whole of your visiting law.

Table 2

When devising this situation the researcher had the standard formulation of 'equivalence relation' and 'partition' in mind (e.g. Stewart and Tall, 2000). And the situation was originally designed with the *intention* of seeing how the students

proceed with what was then considered to be *the only way* of organizing the situation in order to come to *the* definitions of ‘equivalence relation'.

In detail, having captured the reflexivity in the first condition of a visiting law, the situation aimed at leading students to the symmetry and transitivity through creating their own examples demanded in the first task on the one hand, and giving the minimum amount of information demanded in the second task on the other hand.

The study started with a small opportunistic sample of students comprising one graduate mathematics student, two first year mathematics students, two second year physics students (initial sample) , and then with one computer science student, and one sixth-form student (the last two will be used to describe the merits of the study).

The initial data revealed that the students spontaneously created their own way of organizing the given situation which are not necessarily those intended by the situation designer. Accordingly, the *intention* of the study became an investigation of *the ways* that students organize the given situation. In addition, those results led the study to the phenomenographic methods to provide the study with a conceptual framework for describing the variation of ways of organizing the given situation.

Methodology

As mentioned this study adhered to a phenomenographic approach. According to Marton and Booth (1997) phenomenography is a research approach that aims to reveal and describe the variation of ways of experiencing a phenomenon or a situation. Having this in mind, we elaborate our methodology in the context of two interviews with two students having no formal previous experience of equivalence relations and related concepts usually used to define it. Tyler is an undergraduate computer science student and Jimmy is a sixth form student studying mathematics.

The interviews had a simple structure; the two tasks (Table 1&2) were posed in order, but the timing and questions were contingent on students’ responses. The interviews aimed at reaching a mutual understanding between interviewer and interviewee (in the sense of Booth et al, 1999, p.69) of the situation and the ways that interviewee organized it. Therefore the interviewer did not judge the interviewees' utterances as to his own understanding, and insisted on the students giving transparent reasons for their decisions, mainly, as Marton and Booth (1997,p.130) say, “through offering interpretations of different things that interviewee has said earlier in the interview”. Tapes and written work were treated as data; and they analyzed according to the phenomenographic analysis method in which, as Booth (2001, p.172) says, ‘the data is pooled, temporarily losing the individual context in which it was gathered and gaining a collective context of the voices of other individuals who have contributed to the data. The researcher engages with this pool of data and seeks critical differences that can act as catalysts for an understanding of the whole'.

Results

Regarding Jimmy’s work and Tyler’s work, two differences can be identified:

- The difference in what they did to organize the situation
- The difference in their outcomes

The difference in what students did to organize the situation

To satisfy the first condition of the given situation (Table 1), Jimmy and Tyler blacked the diagonal and continued as follows (Table 3):

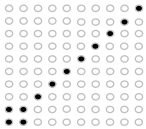
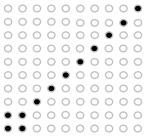
J- Now we have to satisfy the second condition, for each pair of cities, either their visiting-cities are identical, if you have the city one, if you can visit two, you have to, in city two either you can visit city one, like that, you have to because otherwise, they have something in common already, so you have to be able to visit. 	T- If I am in city one, and we allow to visit city two, how the other things need to change, to keep the rules consistent and see either they are completely the same or completely different, so aha, so city two now have to be able to visit city one… 

Table 3

Jimmy "has a rule to apply"; he suspends his reasoning and replicates the result. In other words he replicates a two by two block-square (table 4). On the other hand, Tyler considers two things, "mirroring in y equals x" and "box" (square), and then "to see what was happening" he decides to make city one visit city ten (table 4).

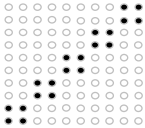
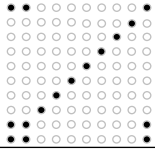
J- And likewise, if you go like that in pairs…It's like paired-up, so if you compare one and two, they have every thing in common, identical, if you compare one and three, one and four, one and five, or one *and six, they have nothing in common…* 	T- … and I realised first that, city ten has to visit city one…so that the second law …city ten has to visit city two…now I look at the city two, now I realised they are different from city one…so I copy number one on to number two also just to keep them the same… 

Table 4

As a result, Tyler abandons the "block square", keeps the "mirroring" and *proves* it as a "general pattern of these dots" (if (x, y) then (y, x)). In addition, the way that he

proves “mirroring”, gives him a new insight, i.e. considering the relationship between any two individual cities:

Tyler- If you allow a city to visit any other city, then it’s gonna end up with having the same visiting-rules as that city that’s allowed to visit and vice versa...

Jimmy still keeps the “block square” to generate his next examples, while Tyler uses “mirroring” and its proof.

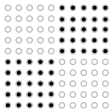
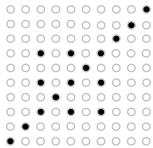
J- …I think there is something to do with square along this line of one and one, two and two, three and three, four and four, five and five; along this line …if you draw a square…people from this city, this city and this city are able to visit each other, they will have identical connection, but other people will not be able to visit them…so people from this group and this group haven’t anything in common, but inside, then, they are identical. 	T- city three can visit city five and seven…so I think of course it’s gonna be reflected in y equals x…No, this is not I want to finish, because now I have cities that have dots in common and they aren’t the same…what I’m missing…what I’m saying here is a sort of square… 

Table 5

Then Tyler draws out, from the big block squares and “a sort of square” appeared in his last example (presented in table 5), the concept of the group of cities:

Tyler- …I completely lost of this sort of way of representing the laws (on grid) because I think they start showing what cities are reachable…in sort of groups you can reach one of the other by travel down the road, you allow to pass the cities between to get from one to other…

Although Jimmy uses the group of cities to organize the given situation, his way is qualitatively different from Tyler's. As it can be seen in the table 4, Jimmy divides cities into two groups, one of them (focal group) includes identical visiting-cities and another one includes the rest (except for his example illustrated in the table 5 in which two views coincide), while Tyler divides cities into groups so that, each group includes identical visiting-cities. That is, from our perspective, Tyler has the notion of partitions, while Jimmy has a split in the set of cities into ‘the group I’m currently working with’ and ‘the rest’.

The difference in their outcomes

While the result of Jimmy’s work is many individual examples, Tyler transcends the situation by introducing new concepts. Particularly, he introduces a new concept with general applicability (the ‘box concept’):

Tyler- How do I say that columns must be the same mathematically? (He writes)

If (x_1, y_1) and (x_1, y_2) and (x_2, y_1) then (x_2, y_2)

Interviewer- Could you explain.

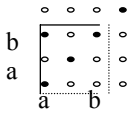
Tyler- I think it's a mathematical way of saying …if a column has two dots, and there is another column with a dot in the same row, then that column must also have the second dot in the same row…I take maybe a box of four dots…I use the coordinate because that makes it very general, and so if I made that my second law, for a mathematician might be easier to follow.

Given this, an equivalence relation can be understood as a relation having the reflexive property and the box property (If (x_1, y_1) and (x_1, y_2) and (x_2, y_1) then (x_2, y_2)). That is, Tyler has explicitly generated a new (and, for us, unexpected) definition (which happens to be mathematically equivalent to the standard definition of equivalence) in order to organize this situation.

As an ongoing study, we are seeking a more detailed picture of such differences in the light of our new data. Thus let us at the moment focus our attention on different ways of organizing related concepts as informed by the present data.

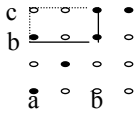
Equivalence relations, revisited

The normative definition of equivalence relation, based on reflexivity, symmetry and transitivity, is widely used to introduce the subject. So let us have a look to the other definitions of it, learned through our data: one based on box concept, and the other based on "triangularity". The following diagrams shows how, having reflexivity and box concept, we can deduce symmetry and transitivity.

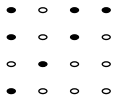


(a, b), (a, a), (b, b) are three corners of the box

(b, a) is the forth corner



(a, b), (b, b), **(b, c)** are three corners of the box



(a, c) is the forth corner

Although the normative way of defining equivalence relations and its definition based on the box concept are logically equivalent, they have dramatically two different representations that could affect students' understanding of the subject. For example, Chin and Tall (2001) suggested "the complexity of the visual representation" as to the transitive law as a source of a "complete dichotomy between the notion of relation (interpreted in terms of Cartesian coordinates) represented by pictures and the notion of the equivalence relation which is not". Accordingly, they suspected that that dichotomy inhibits students from grasping the notion of relation encompassing the notion of equivalence relation. However, the above figures show that the stated dichotomy, to a large extent, depends on the standard way of defining

equivalence relation, i.e. if we define equivalence relation as a relation having the reflexive property and the box property, that dichotomy would disappear.

It is worth saying that the notion of equivalence relation defined by the box concept and its normative definition reveal two different ways of organizing the related concepts. While the former provides us with a simpler visual representation, the latter endows the subject with a seemingly more comprehensive quality in which two important types of relations, equivalence relations and order relations can be seen as particular types of transitive relations. Leaving a concept suitable for organizing a local situation in favour of grasping a more global picture is a particular aspects of mathematics that once again appear as to "triangularity".

Triangularity is the name that for the sake of this paper is given to one of the most common way that our students tackled the situation, i.e. relating the cities in a group of related city without any particular *order* or any *direction*, or referring to equivalent columns without any particular order. Beyond this particular situation, triangularity means when two things are related to a third, the first two are related to each other too. In detail, it is a disjunctive concept, that, if a is related to b and b is related to c then a is related to c *or* if a is related to b and a is related to c then b is related to c (As it can be seen the first part of this or condition is what is known as transitivity). It is the concept that is seemingly behind Euclid's account of equality (far long before having any account of relations or equivalence relations), the first among common notions, that, "things which are equal to the same thing are also equal to one another" (Heath, 1956, p.155). And it is the concept that is clearly behind Freudenthal's account of equivalence relations (in the years of having transitivity as one of the distinct concept comprising equivalence relations). Freudenthal (1966, p.17) defines equivalence relations as a relation possessing the following two properties: first, "every object is equivalent to itself (reflexivity)", and second, if "two objects are equivalent to a third, then they are also mutually equivalent (transitivity)", and shortly after that he notes that those two indicate symmetry property, that, "If an object is equivalent to a second object, then the second object is also equivalent to the first (symmetry)"; but he emphasizes that "actually, the first two properties are sufficient" to define equivalence relations. While, in the course of defining equivalence relations, he uses the term transitivity for what we call triangularity, a few pages on (ibid, p.19), when considering *order* he uses the term transitivity for what is usually known as transitivity:

> …and if, for every three different members a, b, c, of Z it follows from a< b and b< c, that a <c (transitivity of the <-relation).

Having a group of "equivalent elements" in mind, there is no way to separate transitivity from triangularity; that is probably why Freudenthal exploits the term transitivity where he uses triangularity, and in the same vein, Skemp (1971, p. 175) does so:

> The importance of the transitive property is that *any* two elements of the same sub-set in a partition are connected by the equivalence relation.

And that is why no student in our study (not even in the initial interviews where interviewer had a bias toward the standard definition) could notice transitivity as a distinct property. In general, not only in our situation, but also *in any other situation based on splitting a set into disjoint sub-sets by using a particular relation, there is no way to bring the transitivity up unless it is taught.* That is probably why Stewart and Tall (ibid, p.73), *right after* comparing two relations, one splits a certain set into disjoint pieces, and the other does not, "take account of three very trite statements" (including transitivity)as what makes the former work.

Deep down, while by standard account of equivalence relations and order relations, they fall into our hands as special cases of transitive relations, as a drawback, we impose something extra on the equivalence relations, i.e. a sense of direction or order.

Conclusion

As it can be seen in the Mariotti and Fischenbein's study (ibid), it is widely taken for granted that there is a fixed concept that the students are trying to negotiate, but the present study (and implicitly Mariotti and Fischenbein itself) suggests that these open tasks (that designed around an intended concept) can be organized in different ways. Thus a categorization of a) how they are organized and b) what is organized, is of clear value.

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MEDIATION AND INTERPRETATION: EXPLORING THE INTERPERSONAL AND THE INTRAPERSONAL IN PRIMARY MATHEMATICS LESSONS

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This paper describes a theoretical model for examining teaching and learning in primary mathematics classrooms. The model in intended to be both analytical – to provide insights into classroom incidents – and heuristic – to inform planning and practice. This paper reports on the elements of the model, which are based on Vygotskian theory and encompass both the interpersonal and the intrapersonal. An example is provided illustrating how the model can be used to examine the meaning making processes of individual pupils.

1 INTRODUCTION

The focus of this paper arises from interest in how teachers and pupils co-construct mathematical meaning, through the dialectic between the processes of participating in mathematics lessons, and the processes of individual cognition (Cobb & Bauersfeld, 1995). This interest is both theoretical and empirical. Over the course of a five-year longitudinal research programme (the Leverhulme Numeracy Research Programme) a team of researchers has sought ways of developing a theoretical model to help to explain the differential acquisition of mathematics that was observed (see for example, Askew, Brown, Denvir, & Rhodes, 2000).

2 THEORETICAL BACKGROUND

The model is based in a Vygotskian theoretical framework that assumes learning to precede intellectual development through mediated transactions (Cole, 1996; Wertsch, 1991). Matusov (1998) distinguishes between a view of development as coming about through participation or through internalisation, arguing that a preference for one or other of these theoretical positions comes from different world-views. In discussing Matusov's work, Daniels (2001) distinguishes between 'skills and functions in the 'internalisation thesis' and 'meaning' in the 'participation antithesis' (p. 40).

However, rather than preferring one position – participation or internalisation – we consider it is necessary to work with both (whilst sharing other writers' concerns with the use of the metaphor of 'internalisation'). While pupils' participation in mathematics lessons may shape the mathematical understandings that they acquire, it cannot determine then uniquely: one only has to witness the range of meanings demonstrated by pupils who have participated in the 'same' lesson. Hence in the development of a model, we seek to understand both the interpersonal and the intrapersonal and the relationship

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between the two. Other models that influenced our thinking, in particular that of Saxe (1991) incorporated both these aspects, but we sought to develop a model that would also enable us to examine the inter- and intra-personal on their own, as well as in conjunction. Thus there are two parts to the model: an observable set of parameters examining mediating means and a more interpretive set of parameters exploring personal meanings.

3 THE THEORETICAL MODEL: MEDIATING MEANS

At the observable level we found the following four parameters to be the most helpful, in terms of framing both our observations and the subsequent analysis: *tasks*, *artefacts*, *talk* and *actions*.

3.1 Tasks

The mathematics lessons that we observed were all based around a *task* or *tasks* that the teacher initiated for the pupils to work on, and in this, we consider them to be typical of mathematics lessons in England. The teacher herself may have determined the actual nature and content of the *tasks* or she may have directed the pupils to work from a text-book or work-sheet. Whatever the origin, we take *tasks* to be a key mediating means for working with pupils on mathematical meaning. At the observable level, we take *tasks* to be publically set up and initiated, linked, as we argue below, to individuals' sense making; to individuals' *activities*.

3.2 Artefacts

The everyday usage of *artefact* is simply to refer to a material object. While classrooms are clearly full of material objects, our definition of *artefacts* goes beyond 'brutally physical objects' (Bakhurst, 1995). *Artefacts* have a dual nature, having not only a 'material' dimension but also an 'ideal' (conceptual) dimension:

> (T)he artefact bears a certain significance which it possesses, not by virtue of its physical nature, but because it has been produced for a certain use and incorporated into a system of human ends and purposes. The object thus confronts us as an embodiment of meaning, placed and sustained in it by 'aimed-oriented' human activity (Bakhurst, 1995)

In this sense mathematics classrooms are the home of many artefacts: hundred squares, number lines, base ten blocks, etc. Children's fingers or other body parts would also be considered as artefacts when used for counting or calculating. We also consider more transitory objects to be artefacts: symbols, diagrams and so forth. Whilst *talk* could be considered under this definition to be an *artefact*, for reasons set out below, we choose to treat it separately.

Artefacts do not come into being with the 'ideal' in an immediately apprehendable form: the ideality must be mediated, usually through *talk* and *actions*. While a hundred square

could be a material presence for a two-year-old, it would not have the 'ideal' dimension that it might for a ten-year-old. We are interested in how pupils come to 'read into' artefacts the 'ideality' inscribed in the material?

3.3 Talk

As Cole (1996) points out, a 'material' and 'ideal' view of artefacts means that

> the properties of artifacts (sic) apply with equal force whether one is considering language or the more usually noted forms of artifacts such as tables and knives which constitute material culture (p. 117).

Thus we might argue for considering teachers' talk within the category of artefact. However we chose to separate out 'talk' from 'artefacts' for several reasons. Firstly, mathematical classrooms are unusual in the extent to which there are a number of pedagogical artefacts (that are not talk) that exist and take meaning only within the context of classrooms. Many classroom mathematical artefacts are primarily pedagogical and usually only found within classrooms. The artefactual 'paraphernalia' of mathematics classrooms seems worthy of attention its own right.

Secondly, *talk* is probably the principal *artefact* through which teachers and pupils co-construct meanings. Without *talk* classroom participants would have difficulty imbuing pedagogical artefacts or lesson tasks with meanings. *Talk* is not only an *artefact* in its own right: it is also a mediating means for developing shared meanings for *tasks* and other *artefacts*. Thirdly, *talk* is unique amongst *artefacts* in its self-referential nature. Verbal explanations are used to clarify other verbal explanations in a way that physical *artefacts* are not.

Finally, unlike *talk*, the majority of *artefacts* used or produced in mathematics lessons are usually introduced by the teacher. In lessons characterised by a high level of discussion involving pupil-pupil *talk* as well as teacher-pupil *talk* then pupils are involved in the production of mediating means for their peers.

3.4 Actions

Many of the *artefacts* of mathematics classrooms go beyond providing visual images of mathematics and are designed to be acted upon, either by the teacher or pupil or both. Even when not designed to handled (for example a wall mounted 100 square), *actions* are invoked in working with *artefacts* (for example, imagining movement around the squares on a 100 square). Also, in helping pupils appreciate the meaning of mathematical operations that might be associated with symbols, *action* based metaphors are often invoked, verbally and/or through physical models. For example, subtraction as 'take-away' or division as 'share'. In such instances, actions are important mediating means to help link *talk* and *artefacts*.

Although discussion of each of these parameters is treated separately above, they are interdependent. For example the setting up of a classroom *task* may involve use of *talk*, *artefacts* and *actions*.

4 THE THEORETICAL MODEL: PERSONAL MEANINGS

So far we have focused our attention on observable aspects of mathematics lessons. But as Lemon and Taylor (1998) remind us, examining the material world only provides a partial story:

> We never perceive only raw matter, nor do we perceive only mental phenomena. We always experience the action between the two. (p. 230)

Looking at *tasks*, *talk*, *artefacts* and *actions* as mediating means can give some insights into the sorts of experiences in which pupils have the opportunity to participate. However, we also need to consider the sense that participants make of such experiences. While we cannot directly observe such sense making, we can take teachers' and pupils' particular responses to and uses of mediating means as indicators of how they are interpreting their experiences. So linked to our four parameters of mediating means, we have worked with three interpretive parameters of: *activity*, *tools* and *images*.

4.1 Activity

Tasks are publicly set up, *activities* are *tasks* as privately interpreted. In setting up classroom *tasks*, teachers will have their individual interpretations of the *activities* that the *tasks* are intended to provoke (although such understanding may be tacit and the distinction between a *task* and an *activity* not explicitly addressed). Pupils will always interpret classroom *tasks* in the light of their previous experiences and current understandings. However carefully a teacher sets up a *task*, one cannot assume that the individual pupils' interpretations of that *task*, the *activities* that they engage in, are either similar to each other's, or fit with the *activity* expectations of the teacher. Hence the use of the term *activity* to distinguish any one individual's interpretation from the 'public' presentation of a *task*.

4.2 Tools

Cole (1996) argues for the treatment of 'a tool as a subcategory of the more general conception of an artifact' (p. 117). However, rather than consider *tools* as subcategories of *artefacts*, we define them as the personal meanings attached to artefacts.

We begin this definition by drawing on the distinction between interpersonal meanings and intrapersonal meanings and linking these with *artefacts* and *tools* respectively. As indicated we consider *artefacts* as embodying ideas, interpersonal meaning, alongside having some material existence. A hundred square on the classroom wall is intended to be 'aimed-oriented' as embodying particular aspects of the number system. The

interpersonal ‘meaning’ embodied in an *artefact* is, in a sense, objective, transcending the sense making of any particular individual.

On the interpretative plane, we want to argue for a conception of *tools* as more personal arising from the interpersonal meaning unique to any individual working with an *artefact*. Just as the relationship between *tasks* and *activities* needs to be examined, so too the relationship between *artefacts* and *tools*.

4.3 Images

We define *images* as broader than visual images to include verbal or kinaesthetic images. We do not consider that there is any simple one-to-one correspondence between external mediating means and the *images* that might be provoked. For example, an external visual *artefact* may produce an internal kinaesthetic or verbal *image* just as it might produce a internal visual *image*. We are not interested in the mechanisms whereby *images* are provoked, nor in understanding the mechanisims by which external mediating means become internal *images*. We are interested in the impact of individuals’ *images* on interpersonal mathematical meaning making.

Once again, although described separately, these are not independent of each other. The use of particular *tools* may depend on access to certain *images* and together they may affect the nature of *activity*. The following analysis further illustrates the interplay between the parameters.

5 ANALYTICAL APPLICATION OF THE MODEL

As an example of the use of the framework as an analytical tool, we present data from one lesson observed over the course of five-years of lesson observations. (For a more detailed account of the research from which this example is drawn see Brown, 2002). Given the level of detail that the analysis yields, restrictions of space prevent the reporting on the response of more than one pupil; Mayur. However, the full analysis demonstrates that although set up by the teacher to be working on the ‘same’ *task*, the pupils’ individual responses meant that they engaged in distinct *activities* and consequently, we suggest, were likely to have established different meanings.

The group observed, of which Mayur was a member, had not been singled out by the researcher for particular attention, there just happened to a spare chair at the table that they were working at during the observation. After the lesson, the teacher indicated that the children at this table were in the “middle” group of maths attainment. When the researcher (MA) joined them they were each working through calculations from identical worksheets: finding unitary fractions of whole numbers, for example, 1/4 of 36 or 1/5 of 40 (all with an exact whole number answer). There were 12 such calculations for the children to answer.

At the end of the lesson, the children's completed worksheets suggested that the other children in the group had carried out all the calculations correctly whereas Mayur, while getting some correct, had made several errors in his calculations. But as our observations of Mayur's methods of working show, the differences in his answers were not simply the result of correct or incorrect calculations but arose from different interpretations of the *task*: the *activities* that he engaged in changed as he worked through the *task*. (And the full analysis shows that, although arriving at correct answers, neither could the other children be considered to have been engaged in the same *activity*)

Most of the pupils on the table were observed to have chosen one method of calculation and then used that for all the calculations. In contrast, Mayur did not consistently use only one method. He did the first four calculations using a tallying method, for example, finding '1/4 of 36' by marking tallies in rows of 4, each row under the previous one and counting on in 4's until he reached 36 (doing his working on scrap paper that ended up in the waste-bin). Counting the number of rows of tallies gave him his answer. Thus Mayur's initial *actions* involved partitioning the total into requisite groups through the use of the *artefact* of groups of tallies.

However, for the fifth calculation –'1/3 of 21'– Mayur initiated a change of *artefacts* and *actions*. Rather than recording tallies in groups of three as he had done previously, he verbally counted on in threes, keeping track of this by holding up one finger for each multiple of three pronounced. Once he reached 21 he counted the number of raised fingers (seven). Similarly for '1/5 of 30' he counted on in fives, raising a finger for each multiple of five, thus ending up with six raised fingers.

Rather than count in ones (as other children in the group were observed doing) Mayur's initial *artefacts* and *actions* – lining up of the tallies under each other – meant that in a sense his *artefact* controlled his *actions*: once a group of tallies was complete there was an umabiguous signal to start the next set of tallies. This facilitated his use of skip counting in the pattern of multiples. In doing so, he attended to counting up in groups: to a repeated addition method of solution. His use of the pattern of multiples as a *tool* then allowed the development of what were, for him, more efficient methods through different *artefacts* and *actions*, but which in turn appeared to affect his *images* and *activity*.

In coming to represent the divisor as a unitary group – a single finger representing a group, rather than tally all the elements – Mayur employed a different *artefact* (and *tool*), a 'compacted' representation. Instead of relying on paper and pencil to model the full quotitioning of the number to be divided, he only had to count the number of fingers he held up. But with this change of *artefact* there was a resultant shift in his attention and *activity* as his subsequent work demonstrated.

On question seven –‘1/3 of 30’– Mayur announced (to no one in particular) that he was going to ‘cheat’. Looking back at his answer to ‘1/3 of 21’ (7) he immediately held up 7 fingers, counted on in threes from 21 to 30, putting out three more fingers and writing down ‘10’. The next calculation was to find ‘1/10 of 20’, but rather than use tallies or fingers, Mayur immediately wrote down ‘10’.

MA: Why is the answer to that ten?

Mayur: You have to find which table the number is in. Twenty is in the tens table, so the answer is ten.

Similarly he wrote down ‘10’ as the answer to ‘1/8 of 40’ but changed this to ‘4’.

Mayur: I got it wrong. It's not which table it's in, but where in the table.

He wrote down ‘5’ as the answer to ‘1/2 of 50’.

MA: Why is that five?

Mayur: It’s in the ten times table and it’s the fifth one’.

Finally, checking back over his work, Mayur changed his answer to ‘1/3 of 30’ from ‘10’ to ‘3’ and ‘1/10 of 20’ from ‘10’ to ‘2’!

Mayur’s attention to the *action* of holding out the total number of fingers was, initially, correctly linked to the divisor. However, in counting up to the dividend he focused (literally and metaphorically) more on the number of fingers than on the running total. A shift from ‘count on in 4’s to 36’ to ‘9 makes up 36’ (with the 9 being linked to groups of 4 becoming more tacit). So rather than starting with the divisor his attention shifted to the dividend as being the most significant item of information. In attempting to be even more efficient, he began to attend first to the dividend and his *activity* become one of ‘spotting’ the obvious table that that particular dividend would be in. Given, say, a multiple of 10, then the calculation, for him, must be something to do with the ten times table.

Note that initially Mayur went as far as re-interpreting the *task* as the *activity* of ‘which table is this number occurring in’: if the dividend was 30 then the answer must be 10 (30 is obviously in the ten times table). Subsequently he ‘self-corrected’ himself to ‘where in the table’ the dividend was, but still a ‘table’ of his own determination (the most obvious one) rather than determined by the divisor: if the dividend was 30 then the answer must be 3 (it’s the third multiple in the tens table).

6 DISCUSSION

With the *action* of drawing up tallies Mayur, was still producing *artefacts* that represented all the information in the calculation: the size of each group, his progress towards the total number to be divided and when the total was reached. But in the move

to holding up fingers, the only external *artefact* then available was the total number of groups. All the other information Mayur had to hold in his head. Along with focusing on the number of fingers he had held up, his attention shifted away from building up to a total (by skip counting) and instead to focus on the position of the divisor in the pattern of multiples. So while his *tools* may have been more efficient, they influenced his *activity*.

We are not suggesting that alternative *activities* are ones that the children may consciously develop, but that different *activities* are potentially present through the choice of diiferent *artefacts* (for example, columns, fingers) as mediating means and consequently the *tools* children use to carry out the task. The relationship between tools and activity is dialectical – each is informed and informed by the other. It is not simply a case of children understanding *tasks* and then selecting appropriate *artefacts* to use, that each of these together influence the *activities* and *tools* and hence the mathematical meanings. In further work we are examining the role of *talk* in these processes.

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BEING SENSITIVE TO STUDENTS' MATHEMATICAL NEEDS: WHAT DOES IT TAKE?

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This study is about student teachers using questioning to explore the mathematical reasoning of secondary school students aged between 11 and 14 years of age. The student teachers involved were first year students following a four years' initial teaching training course in Malta. The Teaching Triad developed by Barbara Jarowski was used to analyze the students' reports about their questioning. Another aim of the work is to provide the student teachers themselves with a reflective tool for analyzing their own questioning.

THEORETICAL CONSIDERATIONS

The Teaching Triad, developed by Jarowski from an ethnographic study of investigative mathematics teaching, provides a useful method for analyzing teaching. The decisions inherent in teacher discourse are regarded as a complex interplay of decisions of three types; Management of Learning (ML), Student Sensitivity (SS) and Mathematical Challenge (MC). This model was later used to analyze the classroom interactions of other secondary school teachers (Jarowski & Potari, 1998) as well as those of university tutors working with their students (Jarowski, 1999; Jarowski, 2002). In secondary schools, the emerging characteristics of effective teacher interventions involved a harmonious balance of SS and MC interventions. Through SS, the teacher is not only sensitive to the affective responses of the student (SSA) but is also sensitive to the students' cognitive needs (SSC). In providing mathematical challenge, the teacher is prompting students to engage in mathematical thinking and possibly to develop such thinking. This model thus takes into account the social aspect of the teaching/learning situation to include the important role of the teacher in directing such learning. Attention is directed to teachers' planning and to their interventions intended to extend students' mathematical thinking as well as those intended to sustain their interest. As it were, the teacher can be thought of tapping the students' zone of proximal development as described by Vygotsky (1978).

> What then is defined by the zone of proximal development, as determined through problems that children cannot solve independently but only with assistance? The zone of proximal development defines those functions that have not matured but are in the process of maturation, functions that will mature tomorrow but are currently in an embryonic state. These functions could be termed the "buds" or "flowers" of development rather than the fruits of "development". (p. 86)

Treating the teaching/learning interface from the Formative Assessment perspective, Wiliam (1998), argues that support given to learners constitutes formative assessment only if the following five conditions are met:

1. a mechanism exists for evaluating the current level of achievement;
2. a desired level of achievement, beyond the current level (the *reference* level) is identified;
3. there exists some mechanism by which to compare the two levels, establishing the existence of a gap;
4. the learner obtains information about *how* to close the gap;
5. the learner *actually uses this information* in closing the gap. (p.3)

While both belonging to a tradition of socio-constructivist research, the Teaching Triad model and Wiliam's conditions for formative assessment focus on different aspects of the teaching/learning situation. Jarowski's model accounts for an overall view of the teaching/learning situation, emphasizing the need for teachers to plan for learning, to be sensitive to their students' cognitive and affective needs and to offer them mathematical challenge. On the other hand, Wiliam's conditions zoom in on student sensitivity and mathematical challenge and explain how through student sensitivity cognitive decisions (SSC), the teacher helps provoke students' engagement by providing appropriate mathematical challenge. Finally, Wiliam considers that formative assessment also includes that the learner uses such challenge to extend his/her achievement.

THE PRESENT STUDY

The quality of teachers' questioning is undoubtedly one of the crucial factors affecting the quality of pupils' learning. While in a teaching/learning situation, the use of questions can serve a multitude of purposes, the focus of this paper is limited to their use in evaluating pupils' thinking. In this study the use of student teachers' questioning when working in a one-to-one interviewing situation is explored using the Teaching Triad as an analytic tool. This work aims to articulate the strengths and weaknesses of different episodes of questioning intended to evaluate pupils' thinking. The student teachers themselves could also benefit from being involved in this method of analysis, in that they could use it as a tool in reflecting about their questioning.

The data used in this paper comes from an assignment given by the author to first year student teachers following a four-year initial teacher education course. These were asked to give a written test to a secondary school pupil[1] aged between eleven and fourteen years. The student teachers were to use one of the tests on Fractions,

[1] Although the word "student" may be more appropriate here, in this paper, the secondary school students are being referred to as "pupils". This is done so as to distinguish more clearly from the term "student teacher".

Algebra or Measurement produced by the CSMS (Concepts in Secondary Mathematics and Science) team based at Chelsea College, University of London (Hart, 1981; Brown et al, 1984). After correcting their pupils' work, the student teachers were to interview their pupils on some of the test items with the aim of exploring their pupils' mathematical reasoning and establishing, where possible, two gaps in their pupils' understanding of the topics concerned. The student teachers were further directed to audio-tape their interviews and to write a report about their work. Emphasis was made that they were meant to probe into their pupils' reasoning and that they were not being asked to teach their pupils.

The student teachers concerned were following a B.Ed. (Hons) course specializing in the teaching of mathematics at secondary level. In their first year, the focus of the mathematics course is mathematics content at an undergraduate level. During this academic year, they also have a school observation course where they are assigned tasks of a general level that are not specific to mathematics teaching. By this time, they would not have had any formal teaching experience in mathematics as part of their course. For the forty student teachers concerned, the purpose of this work was an assignment following their first methodology course. This fourteen-hour course was delivered by the present author and involved discussions about (i) the nature of mathematics education, (ii) behaviourist theories of learning, (iii) socio-constructivist views on mathematics learning and (iv) talk in the mathematics classroom.

For the student teachers, the idea behind the assignment was to focus on an individual pupil's reasoning prior to setting further activities in an attempt to further his/her mathematical thinking. The whole cycle of a teaching/learning process can be considered as including all the five of Wiliam's conditions for formative assessment cited in the previous section. On the other hand, the student teachers' work was limited to the first three of these. The test and the interview constitute the mechanism for determining the current level of achievement as well as identifying a desired level of achievement beyond the current level. The student teacher was then to use this information in order to describe more fully, where possible, two gaps in the pupils' knowledge.

In analyzing the excepts provided by the students, use was made of the Teaching Triad. Since the purpose of the students' work was to explore their pupils' reasoning, the analysis is focused on the student sensitivity exhibited by the student teacher in the cognitive and affective domains (SSC and SSA). Three excerpts from the student teachers' interviews are discussed in the next section.

RESULTS

Case I

When the student teachers were given the assignment, it was emphasized that the aim was to sort out their pupils' thinking and for this reason, it was important for them to refrain from prompting the correct answers themselves. Still, some students ended up

prompting their pupils in this manner. A number of student teachers were lured into this strategy and after handing their assignments, a few students also claimed that they were tempted into prompting their pupils to get the right answer and found it very difficult to refrain themselves from doing so. One particular example is taken from Caroline's work while she was interviewing Kathy on question 4 of the Algebra test (see Fig. 1). As with all other names used in this paper, Kathy and Caroline are not the real names.

4(ii).	n multiplied by 4 can be written as $4n$. Multiply each of these by 4:	
8	$n + 5$	$3n$
$\underline{32n}$	$\underline{4n + 5}$	$\underline{12n}$

Fig 1: A reproduction of Kathy's written response to qn 4(ii)of the Algebra test

From the written script, Caroline noted that Kathy multiplied 8 by 4 and $3n$ by 4 correctly but failed to get the correct answer in multiplying (n + 5) by 4. In fact, Kathy's answer to this multiplication was $4n + 5$.

1 **I:** How did you work it out?
2 **K:** Uff, I got confused there. See…4 times 8 equals 32…not so?
3 **I:** Yes, go on. How did you work the one with $3n$?
4 K: $3n$ times 4 equals… so 3 times 4 is 12 and there is n as well. So there is 12
5 times n equals $12n$… yes?
6 **I:** Ehe. What about the one in the middle?
7 K: Well, $n + 5$ times 4 is $4n + 5$. Not so?
8 I: If you had the first and last one (before multiplying by 4) what do you have?
9 K: So the 8 and $3n$ plussed?
10 **I:** Ehe. What happens?
11 K: Not $8 + 3n$?
12 I: So $8 + 3n$ is the same as $3n + 8$, no? So if we do $3n + 8$ times 4, aren't we
13 doing ($3n$ times 4) plus (8 times 4).
14 K: Ehe, so the answers are added … giving $12n + 32$.
15 I: Ok, so you are saying that $3n + 8$ times 4 give…?
16 K: $12n + 32$.
17 I: Right. So when we had $n + 5$ times 4 – isn't it like you had (n times 4) +
18 (5 times 4)?
19 K: Yes, I see. I understood! The answer is $4n + 20$
20 **I:** Fine. Well done!

This excerpt in fact confirms that one particular gap in Kathy's reasoning is the need for her to appreciate the property of distributativity of multiplication over addition, a

gap that was already apparent from the correction of Kathy's script. The interview yielded nothing new about the cognitive meanings Kathy attaches to these algebraic expressions. Instead, Caroline gave Kathy a rule that could be applied in similar problems- a rule that in no way connects to other mathematical ideas. The affective response in the last two lines of the excerpt similarly reflect that both Caroline and Kathy were pleased with this outcome. Implicitly, the student teacher is here confirming what Kathy has already learnt, namely that getting the answer right is what counts. The uncertainty in Kathy's first comment, "Uff, I got confused there", remains unexplored. Why was she confused? Was she simply unsure of the answers or of some of them? Overall, Kathy is not being challenged to engage in any mathematical thinking further than the procedural thinking involved in knowing how to act in a similar situation.

Case II

In a number of student teacher assignments, the verbal prompts given by the student teachers suggest that the pupils' initial method of solution of the set problems were completely disregarded by the student-teacher. A case in point is that of Stefan's assignment who gave the Fractions Test to Maria. Stefan chose to interview Maria on question 24 whose written solution to this question is shown in Fig. 2. The excerpt from the interview regarding this question follows.

24. A relay race is run in stages of $\frac{3}{4}$ km each. Each runner runs one stage. How many runners would be required to run a total distance of $\frac{1}{8}$ km?
$\frac{1}{8}$ $\frac{1}{8}$ $\frac{1}{8}$

Fig. 2. A reproduction of Maria's written response to question 24

I: Maria, look at this question (pointing to question 24 now. Why did you write $^1/_8$, $^1/_8$ and $^1/_8$?

M: Heq… because $^1/_8$, $^1/_8$ and $^1/_8$ and …emm, I don't know!

At this point in the assignment Stefan comments that it was evident that Maria was really confused and thus he decided to help her build her confidence. The interview continues:

I: Maria, How many Lm2 (two maltese liri) are there in Lm10? (Stefan also wrote down the problem for Maria on a piece of paper).

M: Lm2, Lm2, Lm2, Lm2 and Lm2 (using her right hand to count). There are 5 in all.

I: Very good Maria, now explain to me how you can do this using multiplication or division

M: Heq, 10 divided by 2 and the answer is 5.

I: Yes, that's right, Maria! … If now you have Lm $2^1/_2$ and Lm $12^1/_2$ instead of Lm2 and Lm10 (Stefan rubbed out the Lm 2 and Lm10 on the problem)

13 M: The same… Lm12$^1/_2$ division by Lm2$^1/_2$.

14 I: That's right. So if you need to find the fraction made by a quantity of another

15 quantity you just need to apply the normal operations we use for 1,2,3,4,

16 …etc. Therefore in question 24, you just need to do $^3/_4$ division by $^1/_8$.

17 M: Yes! I was going to do it like that, that's why I wrote $^1/_8$, $^1/_8$, $^1/_8$ … but then

18 I got confused.

The first comment is about Stefan's student sensitivity as far as the cognitive is concerned. Maria's written response suggests that she was attempting repeated addition to solve the question (see Fig.2). Similarly, Maria's initial response to the to the whole number division problem (line 6) posed by Stefan follows the same strategy. By shifting away from Maria's method towards the more formal division method, Stefan is closing down on the possibility of exposing why Maria got stuck with the repeated addition method when working on the fractions problem. Instead, as far as Maria's mathematical processes are concerned, the interview does nothing more than to indicating even more strongly, that Maria's initial strategy to question 24 was in fact the use of repeated addition.

On the affective side, on seeing that Maria got confused, Stefan took over and gradually prompted "his" method for a solution. The stress that arose when Maria got confused is very understandable and it is clear that if one would like to explore Maria's thinking more fully, it is necessary to learn how to handle such stress and utilize it in a positive manner. The next excerpt provides a clue on how this can be done.

Case III

3. A piece of ribbon 17cm long has to be cut into 4 equal pieces.

Tick the answer you think is most accurate for the length of each piece.

(a) 4 cm, remainder 1 piece

(b) 4 cm, remainder 1 cm

(c) **4**$^1/_4$ cm

(d) $^4/_{17}$ cm

Fig 3: A reproduction of Pauline's response to qn 3 of the Fractions paper.

In this excerpt, the interviewer, Sandra is asking twelve-year old Pauline about her written response to question 3 of the Fractions paper. (see Fig, 3).

1 I: Now… how did you work out number 3? You can use rough work if

2 you like.

3 P: The question… A piece of ribbon 17cm long has to be cut into 4 pieces. When cut, I think it comes to … (*points to* $^4/_{17}$ cm) four over seventeen.

4 I: But why? Did you try to work it out?

5 P: … Boqq… I forgot how I did it
6 I: Ok, don't worry… try it again.
7 P: So, since I saw these two numbers (*points to 4 and 17 in the question*),
8 … I thought this was the answer (*points to* $^{4}/_{17}$ cm).
9 I: Ok.
10 P: But if you think it out, it is four… you take the four times table, it is sixteen
11 not seventeen. So it is 4 cm remainder 1.
12 I: Now do you think that 1 cm can be broken down into 4 pieces in some way?
13 P: Eh…yes
14 I: How long would each piece be?
15 P: 0.2
16 I: Try to add 0.2 for four times.
17 P: You can cut out 0.2,0.2,0.3,0.3. … *interview continues.*

In this case, the interviewer, Sandra is not prompting Pauline with an alternative method. There is an initial negative response from Pauline (line 5) when she says "…boqq… I forgot how I did it". Here Sandra does not give up on unfolding Pauline's ideas who reveals that she was in a sense guessing at an answer. The mere "O.k." (line 9) in Sandra's response implies an acceptance of Sandra's thinking. This served her to think further and to come up with a more meaningful answer in her next response (lines 10-11). Sandra's next question (line 12) again works with Pauline's earlier construction of the string as 4 pieces of 4cm and 1 cm left over. Later on in the interview (not included here), Sandra discovers another of Pauline's difficulties; she could not work out 0.5 ÷ 2.

DISCUSSION

The high incidence of student teachers' prompting of the 'teacher's method' calls for comment. This is especially significant given the emphasis made that once the assignment called for exploration of the students' reasoning, they were to refrain from prompting a solution to the set questions. There may be various reasons why prompting is so persuasive.

For one, in their own learning experiences, these student teachers would have been heavily exposed to the traditional transmission method or the 'teaching by telling method' (Seegers & Gravemeijer, 1997). This behaviourist approach rests on the belief that learners are passive knowledge receivers and need to be told. Consequently the questioning is not directed towards what the pupils already know but at what is considered that they should know. Another reason for prompting could be that the student teachers themselves were not sufficiently flexible in their mathematical thinking to allow them to recognize different possible methods of

struggling with the set questions. This is particularly relevant in this case because of the student teachers' inexperience of teaching.

A third reason emerges from the results of the three interviews discussed in this paper. Unlike the previous cases, there was no evidence of prompting in Case III. This interview stands out in the affective responses of the interviewer. She reacted in a very positive way to the pupil's frustrations and accepted that confusion and getting incorrect answers is part of the learning process. The focus was not that the pupil gets the correct answer at each stage, but rather that the pupil engages in thinking about her work. In short, she was showing profound respect for her pupil's mathematical thinking.

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A NEW PRACTICE EVOLVING IN LEARNING MATHEMATICS: DIFFERENCES IN STUDENTS' WRITTEN RECORDS WITH CAS

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Students who learn mathematics with CAS calculators are likely to develop new practices for doing and recording mathematics. Students discussed in this paper were able to use CAS calculators in examinations, making their own decisions about what to document as written records for solutions to problems. A comparison of some features of written records produced by these students, with an achievement matched random sample of students with only graphics calculators, gives insight into the new practice which is emerging. Students who had learned with CAS wrote generally shorter answers, used more ordinary words and used function notation more frequently but they did not over-use non-standard calculator syntax.

INTRODUCTION

When students learn mathematics with CAS calculators and can use it in examinations, they are likely to use a combination of CAS and pen-and-paper techniques, making decisions about which is more efficient based on previous experiences and personal competency with CAS and pen-and-paper techniques. Because intermediate steps in routine procedures carried out with technology are not available for inspection, students who use CAS cannot provide the reader (including an unknown examiner in a high-stakes examination) with the traditional form of written record of solution. A new mathematical practice is therefore likely to evolve in this situation. This paper reports a study of this evolving practice, by documenting four features of the written records provided in a high-stakes university entrance and final school examination, by a group of students using CAS and comparing them to a matched group of students not using CAS.

There is little relevant literature. The strong literature related to argumentation and communication in mathematics classrooms (for example, Yackel, 2001; Krummheuer, 1995) has a focus on how students communicate mathematical thinking during classroom interactions, but is overwhelmingly concerned with verbal communication, dialogue and interaction. Our concern here is different because it deals with written communication of mathematical thinking and also because it is concerned, not with the process that students went through to solve problems, but with the end product, the written record, which is used to communicate a mathematical solution, in this case to an examiner. Literature dealing with the effects of CAS on teaching, learning and examinations has also not considered written records. An important thread in this literature (see for example, Artigue, 2002; Guin and Trouche, 1999; Pierce and Stacey, to appear) deals with the development of effective use of CAS in the classroom, but not how students communicate

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mathematical thinking once CAS is used to solve problems. Similarly the literature related to assessment with CAS (see for example, Flynn, 2003; Kokol-Voljc, 2000) considers required changes to assessment items when intermediate steps are not available, but not how the responses should be written.

This paper will analyse four features of written records for selected problems in the 2002 Year 12 externally set and marked Mathematics examinations in Victoria, Australia. The written records were produced by two cohorts of students, one that learned mathematics with CAS calculators and one that learned mathematics with graphics calculators. The CAS students were from three schools offering *Mathematical Methods (CAS),* a new subject (Victorian Curriculum and Assessment Authority, 2001) offered in Year 12 for the first time. The students had used CAS in both 2001 and 2002. Further details of their program and learning are available from the project web-site (www.edfac.unimelb.edu.au/DSME/CAS-CAT) and Stacey (2003). The other students were undertaking the standard subject *Mathematical Methods* (VCAA, 1999). Examinations in both subjects were externally set and graded, with a number of common questions.

Ball (submitted), a revised version of her 2003 CAME paper (see http://ltsn.mathstore.ac.uk/came), has previously reported differences in the written records of the CAS and non-CAS students for one of the common examination questions (Question 1b). She found that CAS written records tended to be shorter on average than non-CAS written records and that CAS written records contained more words than non-CAS written records. For example, Question 1b involved solution of two simultaneous equations and more than 40% of CAS written records (n=78) contained the word 'solve' while only one non-CAS written record (n=78) included this word. CAS written records also contained more function notation than non-CAS written records and there was evidence of some non-standard notation that could be directly linked to a CAS entry. From the analysis of Question 1b, it appeared that CAS students were developing a practice for writing mathematical solutions that had a number of differences to the practice being observed in the work of the non-CAS students. This paper will consider two more questions from the same examination and carry out a similar analysis to investigate whether the differences in written records observed in Question 1b are apparent in other questions.

CAS AND NON-CAS STUDENTS

During 2002 the students undertaking year 12 *Mathematical Methods (CAS)* learned mathematics with a TI89, HP40G or CASIO FX 2.0 CAS calculator and use of CAS was unrestricted in the final examination. These students are referred to as CAS students and their scripts as CAS scripts (see Table 1). Note that this is regardless of whether the student actually used CAS in the solution being analysed. Mathematical Methods students learned mathematics with a graphics calculator and could use it without restriction in the examination. These students are referred to as non-CAS students and their scripts as non-CAS scripts. Differences are summarized in Table 1.

	CAS student	Non-CAS student
Examination script	CAS script	Non-CAS script
Subject studied	Mathematical Methods (CAS)	Mathematical Methods
Technology used to learn mathematics	CAS (TI89, HP 40G, Casio FX 2.0)	Graphics calculator (same brands)
Techniques available for solving problems in examinations	Pen-and-paper Technology-(Graphical, Numerical and Symbolic)	Pen-and-paper Technology-(Graphical and Numerical)

Table 1: Comparison of two cohorts of students

CAS students were familiar with a rubric designed to guide practice for writing solutions in a CAS classroom. The RIPA rubric (Ball and Stacey, 2003) was created in response to students' and teachers' needs. RIPA promoted use of mathematical notation rather than calculator syntax and the recording of reasons (R), information and inputs (I), a plan for the solution path (P) and some answers (A) in written records. If students include reasons, a plan and calculator inputs (using mathematical notation) then we expect more words in students' solutions. All teachers in the research project stressed the importance of clearly communicating written records.

SAMPLE EXAMINATION SCRIPTS AND QUESTIONS ANALYSED

The sample written records to be discussed are from the entire 78 Mathematical Methods (CAS) "examination 2" scripts and a stratified random sample of 78 Year 12 Mathematical Methods "examination 2" scripts. The random sample of non-CAS scripts was matched to the achievement of the CAS scripts, as the purpose of this paper was not to compare the relative achievements of the two groups.

Questions 3i and 3ii (see Figure 1), common to both the CAS and non-CAS examination, are discussed in this paper and compared to results from the initial analysis (Ball, submitted) of Question 1b. Students needed to provide reasoning to show two given results and find the coordinates of a point of intersection of two graphs.

VCE Mathematical Methods (CAS) Pilot Study Examination 2 (abbreviated questions)
Q 1:… According to Fitts' Law, for a fixed distance traveled by the mouse, the time taken, in seconds, is given by $a - b\log_e(x)$, $0 < x \leq 5$, where x cm is the button width and a and b are positive constants for a particular user…
Q1b. Mickey decides to find the values of a and b for his use. He finds that when x is 1, his time is 0.5 seconds and when x is 1.5, his time is 0.3 seconds. Find the exact values of a and b for Mickey.
Q 3: The diagram shows the curve whose equation is $y = \frac{1}{2}(2x^4 - x^3 - 5x^2 + 3x)$ and the normal to the curve at A where x=1.[Graph shown with intersections at A and B]
Q3i. Show that the equation of this normal is $y=x-1.5$.
Q3ii. Show that this normal is a tangent to the curve at point B and find the exact values of the coordinates of B.

Figure 1: Examination questions analysed (Q1b, Q3i, Q3ii) (VCAA, 2002)

Analysis of these two problems enabled investigation of whether observed differences in features of written records for Question 1b were also apparent in the written records for these additional two common questions. This provides some insight into whether or not features of CAS written records appear as part of a new practice that students have developed for recording mathematical thinking or just in response to particular problems.

Four features of written records

Following Ball (submitted), the use of words and mathematical notation in students' written records are categorised in two ways, as shown in Table 2. The length of written records was also measured, simply as the number of lines on the page that contained any working. It was also noted whether solutions used function notation (i.e. $f(x)$). As an example, the length of the written record in Figure 2 was seven. The line showing $y = \frac{1}{2}$ follows from the statement x=1 which shows that it is a separate line. The record is categorised as M′ as it contains evidence of non-standard notation. The student has used $|$ to indicate substitution and also given the CAS syntax $\text{solve}(y - (-\frac{1}{2}) = 1(x-1), y)$ in solving for *y*. The solution is classified as W because it contains words (importantly "solve"). It does not use function notation. More sophisticated measures and definitions did not seem to give different results to these simple ones and were harder to implement consistently.

Code	Written record
M	contains standard mathematical notation only
M′	contains some non-standard mathematical notation
W	contains one or more words that can be found in a dictionary
W′	does not contain any words that can be found in a dictionary

Table 2: Codes for categories of written records (Ball, submitted)

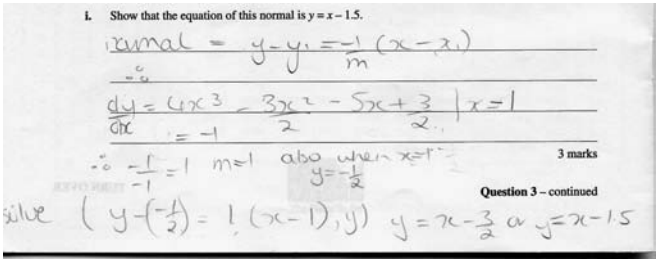

Figure 2: Example of written record containing non-standard notation

RESULTS AND DISCUSSION

Use of standard and non-standard mathematical notation

There were few instances of non-standard notation evident in the written records for the two problems (see Table 3). Q3i and Q3ii each had two occurrences of non-standard notation in CAS solutions and Q3ii had one instance of non-standard notation in a non-CAS solution when a student recorded the name of the calculator program (FCTPOLY2) used to factorise a polynomial. There is no statistical difference between CAS and non-CAS students (χ^2 corrected = 0.44, d.f. = 1, p = 0.506). Non-standard notation given in CAS written records was in the form of a CAS instruction for solving or for substitution, both of which are shown in Figure 2. This limited use of non-standard notation was also reported by Ball for Q1b.

Features of Solutions	Solutions to Q3i		Solutions to Q3ii	
	CAS	NonCAS	CAS	NonCAS
M+W	64	65	65	58
M+W'	9	12	6	15
M'+W	2	0	2	1
M'+W'	0	0	0	0
Written record contained:				
‘solve’	4	0	22	1
‘substitute’	16	7	20	7
‘simultaneous’	0	0	1	0
‘Define’	1	0	0	0
‘and’	4	5	8	5
‘equation’	17	21	26	10
function notation	13	10	15	9

Table 3 Number of solutions exhibiting specified features of written records

The results for the two questions discussed in this paper and previous analysis of Q1b suggest that most students are careful to use standard mathematical notation in recording their solutions even though they have learned mathematics with CAS and may have used CAS for various steps within solutions. In class, the non-standard notation we observed was generally of the nature of generic (not brand-specific) CAS syntax and as such is easily understood by someone with a mathematical background. We expect that some of these currently non-standard notations may become standard.

Length of written records

The lengths of written records for Q3i and Q3ii are shown in Figures 3 and 4. Q3i had a number of shorter CAS written records. The average length of written records for each question (n=78 for each) was found to be significantly greater for non-CAS students in Q3i (CAS 5.5, Non-CAS 8.3, t=6.4, d.f.= 154, p=0.000) but almost the same for the two groups of students in Q3ii (CAS 6.9, Non-CAS 7.0). For Q1b

(n=78) the average length was found to be greater for non-CAS written records (CAS 5.2, Non-CAS 7.2). If we only consider written responses of students that responded to the task (i.e. not length 0), the same results still hold. These results reflect the fact that CAS students are able to perform intermediate routine steps with CAS. This is an important consideration for teachers and students because in these examinations students need to communicate enough appropriate working to access intermediate marks if their final answer is incorrect. The CAS students in this study were aware that examiners needed to be able to follow the working documented in written records. This sort of consideration will mould evolving practice.

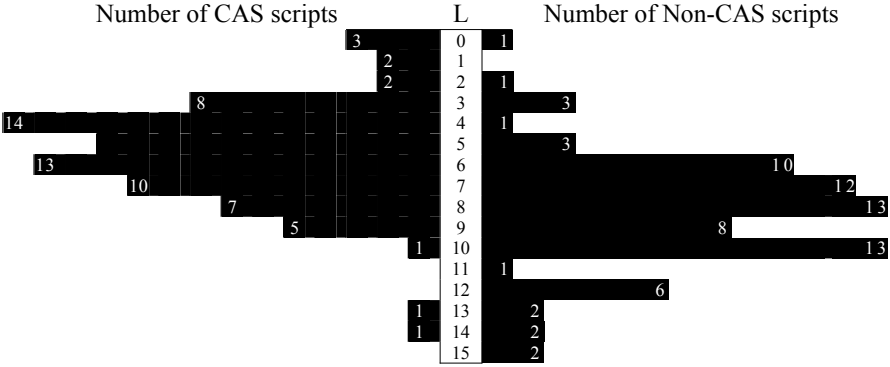

Figure 3 Lengths (L) of CAS and non-CAS written records for Q3i

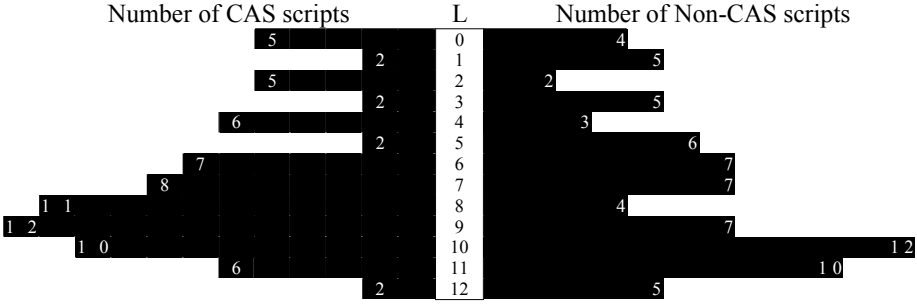

Figure 4 Lengths (L) of CAS and non-CAS written records for Q3ii

Use of words

For Q3i and Q3ii, the percentage of written records containing words was greater for students who learned mathematics with CAS than for non-CAS students. Combining the data in Table 3 for both questions shows 89% of CAS solutions and 82% of non-CAS solutions contained words, a notable overall difference (χ^2 corrected =3.10, d.f.=1, p=0.078). Overall, the high usage of words and in particular use of the word "solve" would suggest that access to CAS is impacting on the written records and

possibly also the way in which students think about mathematical commands. The words used in CAS and non-CAS records had interesting differences. Non-CAS written records tended to contain the words “substitute”, “and”, or “equation” – this is familiar to us, but we had not noticed the restricted range of words until this study. Surprisingly, use of the word “solve” was mainly observed in CAS written records with nearly 30% of CAS written records for Q3ii including this word but few non-CAS records. The use of “solve” in CAS written records could be attributed to a number of factors. We have observed that students often write this word when they are recording CAS syntax. We propose that these students may be thinking about solving at a more global level than non-CAS students. Non-CAS students need to attend to the intermediate steps of a solution, rather than thinking about “solving” overall. For example, to solve a quadratic equation they may first consider rewriting it so that they have a quadratic expression equal to zero, then make an attempt to factorise the quadratic expression and so the focus would be on factorizing rather than thinking about the overall process which is solving. A CAS student can just recognise that they need to solve.

Use of function notation

CAS written records also contained more use of function notation than non-CAS written records for both questions (see Table 3), although the difference is not statistically significant (χ^2 corrected =1.95, d.f.=1, p=0.163). Given that these results occurred for all three questions, this could indicate a new practice for recording. It probably results from the ease of use of function notation with CAS, and the technical benefits of defining functions explicitly for subsequent solving, substituting etc. CAS students may get into the habit of recording a function using function notation initially in their written records to facilitate later use of the inbuilt “define” feature in the solution, and hence access to simplified CAS input.

CONCLUSION

The analysis of an additional two questions generally supports the observations of Ball (submitted) that CAS solutions will generally be shorter than non-CAS solutions, that they will contain more words and use function notation more. Some changes may occur in the mathematical notation that is regarded as acceptable, but there need be no fears that students will replace standard notation by incomprehensible machine-speak. There is evidence that a new mathematical practice is evolving with this tool. Teachers and others need to actively guide this evolution in desired directions.

Acknowledgement

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EMPOWERING ANDREA TO HELP YEAR 5 STUDENTS CONSTRUCT FRACTION UNDERSTANDING[1]

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This paper provides a glimpse into the positive effect on student learning as a result of empowering a classroom teacher of 20 years (Andrea) with subject matter knowledge relevant to developing fraction understanding. Having a facility with fractions is essential for life skills in any society, whether metricated or non-metricated, and yet students the world over are failing in this aspect of mathematics (Queensland Studies Authority, 2002; TIMSS, 1997). Understanding fractions requires comprehension and coordination of several powerful mathematical processes (e.g., unitising, reunitising, and multiplicative relationships) (Baturo, 1997, 2000). While this paper will report on student learning outcomes, its major focus is to tell Andrea's story and from this to draw implications for pre-service education and teaching.

INTRODUCTION

Andrea, who was teaching in a small rural town in Queensland, was a brilliant practitioner. Using Askew, Brown, Denvir, and Rhodes's (2002), parameters for evaluating classroom interactions; I rated her at the highest level in each of the four parameters (Tasks, Tools, Talk, Expectations and Norms). Andrea had developed a truly remarkable community of mathematics inquiry where interest and engagement were often sustained in thought-provoking tasks in whole-class situations. Her principal, colleagues, and parents all attested to her teaching credibility.

I was privileged to observe an impromptu discussion which was initiated by a student who asked which was larger in value – 0.3 or 1/3. Andrea's response was to write both numbers on the board and ask: *Can we tell by just looking at the numbers*? The students replied in diverse ways; some said yes; some said no, one said $^1/_3$ is bigger because *its bits are bigger*, while another said $^1/3$ is bigger because *you can put it into ninths*. Andrea didn't tell the latter student that his thinking was inappropriate but drew, on the board, a rectangle which she partitioned into ninths, asking: *What has this been partitioned into? What can I do to show thirds?* [By doing this she showed she was looking for reunitising.] She accepted responses from students but validated them with the remainder of the class, thus setting up a debate. Finally, the class agreed on appropriate shading to show 1 third, which Andrea followed by asking: *Does this help us compare 1 third with 3 tenths*? Discussion drew from the students that "the same" (congruent) wholes would be needed to compare thirds and tenths. Only then did Andrea throw the problem back to the students to either work alone or in groups to come up with a solution.

[1]This study is part of a much larger study supported by the Department of Education, Science and Training, Canberra, Australia.

The students were "riveted" by Andrea's style of questioning and were eager participants in the discussions. No child was made to feel embarrassed by any response s/he made; in fact, Andrea made inappropriate responses the basis for her line of questioning by throwing the responses back to the class for consideration and amendment. However, even though Andrea was an excellent teacher with very good pedagogic knowledge, she was unable to help her students construct *part/whole* fraction understandings that were robust enough to apply to a variety of tasks (see Figure 1) until she, herself, understood the structural basis of the topic and gained specialist techniques that relate to such structural understanding.

BACKGROUND AND THEORY

Fundamental to the *part/whole* fraction subconstruct is the notion of *partitioning* a whole, whatever its representation, into a number of *equal* parts and composing and recomposing (i.e., *unitising* and *reunitising*) the equal parts to the initial whole. According to Kieren (1983), partitioning experiences may be as important to the development of rational number concepts as counting experiences are to the development of whole number concepts. Students, therefore, should be provided with several opportunities to partition a variety of fraction models in a variety of ways so that they come to understand that ½ (for example) always represents one of two equal pieces. Partitioning, unitising and reunitising are often the source of students' conceptual and perceptual difficulties in interpreting rational-number representations (Baturo, 1997, 1999, 2000; Behr, Harel, Lesh & Post, 1992; Kieren, 1983; Lamon, 1996; Pothier & Sawada, 1983). In particular, reunitising, the ability to change one's perception of the unit (i.e., to also see one whole partitioned into 10 equal parts as five lots of 2 parts and two lots of 5 parts), requires a flexibility of thinking that is often too difficult for some students.

Therefore, the secret to constructing a robust knowledge of the part/whole subconstruct is experience with: (a) partitioning and unitising in a variety of situations; (b) the two-way process of partitioning and unitising (i.e., constructing two-fifths when given one whole and constructing one whole when given two fifths); and (c) examples of wholes and partitioning that require reunitising (i.e., constructing three-tenths from a whole partitioned into fifths and constructing four fifths from a whole partitioned into tenths). The first of these experiences is based on developing fraction structural knowledge (Sfard, 1991) which requires teaching strategies focusing on the notion of unit/whole, partitioning the whole to form equal parts (Lamon, 1996; Pothier & Sawada, 1983), and repartitioning the whole to form smaller equal parts and to reconstruct the unit (Baturo, 2000; Behr, et al., 1992).The latter two experiences are achieved by the generic teaching processes of *reversing* (Krutetskii, 1976; RAND, 2003) and representing the concepts by both *prototypic* and *nonprototypic* models (Hershkowitz, 1989).

In this paper, I describe and analyse the changes in Andrea's knowledge and teaching practices and in the learning outcomes of her students that were a consequence of Andrea's gaining the structural knowledge and generic techniques above. This is

done in order to unpack the crucial components of this teacher knowledge that had the positive effect on student outcomes. The project from which this paper emerged involved eight other mathematics-education academics and I working with eight schools across Queensland in order to identify those factors that enhance student numeracy outcomes. The schools were chosen on the willingness of three to five of their teachers to be involved in the year-long project, which was essentially a model of professional development-in-practice (i.e., in authentic classrooms). We worked with these teachers in their classrooms on topics that the teachers themselves had identified as problematic. For some teachers, this may have been related to generic pedagogical concerns (e.g., improving mathematics engagement). However, the three teachers in my school each identified aspects of mathematics that their students were failing to understand. Andrea nominated the part/whole fraction subconstruct as the area in which she felt that she was unable to teach effectively.

THE STUDY

The methodology I adopted for this study was mixed method, a combination of quantitative research on student outcomes and qualitative and interpretive research on Andrea and her class as a case study. The subjects were Andrea and her Year 5 students and the Year 5 students of a control (comparison) school that was selected to match as closely as possible the various characteristics of Andrea's class, school and community.

Data gathering methods. The data gathering methods I used were observation, interviewing and testing. To explore the process by which Andrea and her class changed their teaching and learning, I observed and videotaped the classes in which she taught fractions and talked to her regularly. To identify growth in fraction knowledge, Andrea and I administered a fraction pre-test in mid April (beginning of Term 2) and post-test in late June (beginning of Term 3). To determine the effects of the academic-teacher collaborations, Andrea and I administered a standardised test covering Number and Space (Part A) and Measurement (Part B), and a research-developed instrument covering Chance at the beginning and end of the year. The results from these classes were compared with those from a comparison school to identify changes in outcomes above those due to maturation.

I developed the fraction test (see Appendix) because the standardised test was not sufficiently detailed in the domain of fractions to provide insights into Andrea's students' specific cognitive difficulties. I needed an instrument that took cognisance of the major fraction concepts and processes, and the generic processes of reversing and representing with prototypic and non-prototypic models.

Fraction test analysis. Items 1, 2, 3a, 3c, 3e, 4 and 5a were designed to assess the students' ability to identify (unitise) fractions from a variety of representations of the whole (area, set, and linear models). Since the area models had been taught in Year 4 and the set and linear models were novel for these students, I expected that the students would perform worst on the set representations of the whole as many students find it

difficult to hold a discrete set of objects in the mind as one whole. Items 3b, 3d, 3f, and 5b were designed to assess the students' ability to reunitise fractions represented by area, set, and linear models. Items 3b and 5a required the repartitioning of the given equal parts whilst 3d and 3f required a recombining of a larger number of equal parts into a smaller number of equal parts.

Items 6, 7, and 8 were designed to assess the students' ability to construct the whole when given the part (i.e., reversing). All other items had provided the whole partitioned into parts. Items 6 and 7 involved area representations. Although Item 6 involved a unit fraction, this was made more difficult by including two parts to be considered as the unit part. Item 7 was a non-unit fraction and required the students to identify the unit part. Item 8 was quite difficult as it involved a set representation of a non-unit fraction.

Procedure. After the tests had been administered, I met with Andrea to plan her unit and provide her with information for this planning. The lesson plan sequences were developed collaboratively by Andrea and me and then detailed by Andrea (Baturo, Warren, & Cooper, 2003). Emphasis was placed on the need for Andrea to develop her own activities specific to the topic.

Overarching the planning sequence was my (Baturo, 1998) teaching and learning theory which encompasses entry knowledge, representational knowledge, procedural knowledge, and structural knowledge. The use of appropriate materials to help students construct mental models was incorporated and reversing activities (e.g., "This is the whole; what part/fraction is represented?" and "This is the part/fraction; construct the whole.") and nonprototypic representations were included to develop robust knowledge. It was agreed that Andrea would include activities involving 10 and 100 equal parts in order to strengthen her students' understanding of decimal fractions.

RESULTS

I analysed Andrea's students' test responses descriptively and statistically, reread the observation field notes and viewed the videotapes, and combined the data to make a rich description of Andrea's progress.

The collaboration. The first activity was the development of the fraction test to determine Andrea's students' *Entry knowledge* (Baturo, 1998). As soon as Andrea saw the developed instrument, she knew immediately the importance of the reversing and nonprototypic representations, exclaiming: *That's what I've been missing!* Thus. Both the instrument and the students responses (which were poor in significant areas – see Figure 1) became excellent springboards for discussion of the teaching sequences for developing fraction understanding. As a consequence, Andrea spent most of Term 2 almost exclusively on re-teaching the fraction concepts, focusing first on partitioning a variety of prototypic and nonprototypic wholes, and then reunitising area, set, and linear models including nonprototypic representations of the whole and the parts, as well as on reversing activities (i.e., whole→part; part→whole). However, she found that her students were unable to process set models so these were delayed until the end of the year. Her excellent pedagogy skills combined with the more sophisticated

techniques and richer representations that emerged from the joint planning meant that she was able to challenge her students' understanding at a greater depth than before.

The test results. The results of the items and sub-items (see Figure 1) show that the students initially exhibited impoverished knowledge in Items 1, 3b, 3f, 5a, 5b, 6, 7, and 8; however, as Figure 1 also shows, these greatly improved by the post-test.

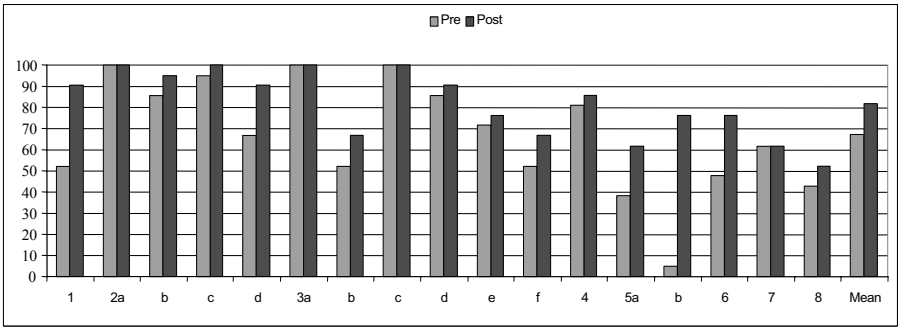

***Figure 1*. Pre and post means for Andrea's Year 5 students with respect to the fractions test.**

The increase in students' outcomes across the collaboration was also evident when I restructured the data according to the major process being assessed. Table 1 provides class and overall means (%) for the unitising items while Table 2 provides means (%) of the items related to reunitising, and reversing.

Table 1: ***Means (%) for All Unitising Items in the Fractions Cognitive Diagnostic Test***

	Unitising Items Individual Students' Means (%) and Overall Mean (%)										
	1	2a	b	c	d	3a	c	e	4	5a	Overall Mean
Pre	52	100	86	95	67	100	100	71	81	38	79
Post	91	100	95	100	91	100	100	76	86	62	90

Table 2: ***Means (%) for All Reunitising and Reversing Items in the Fractions Cognitive Diagnostic Test***

	Reunitising & Reversing Items Individual Students' Means (%) and Class Mean (%)									
	Reunitising Item						*Reversing Items*			
	3b	3d	3f	5b	Class Mean		6	7	8	Class Mean
Pre	52	86	52	05	49		48	62	43	51
Post	67	91	67	76	75		76	62	52	64

Figure 2 shows the pre and post common and decimal fraction means (taken at February and November respectively) with respect to the standardised test administered to all classes in the larger study. Pair-wise comparisons undertaken with

the pre/post results revealed that the increased performance was significant for common fractions (p = 0.000) and for decimal fractions (p = 0.004). The post means also supports the supposition that the learning that occurred in Term 2 was robust for common fractions and flowed on to decimal fraction understanding.

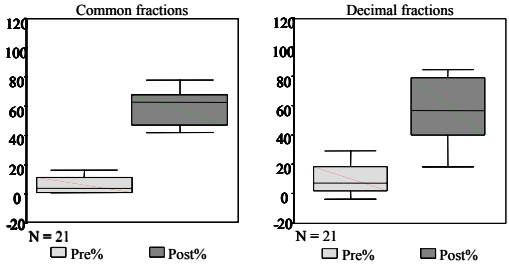

***Figure 2*. Pre and post standardised test means for Andrea's Year 5 class with respect to common fractions and decimal fractions.**

With respect to relative performance on the standardised test with the comparison class, the pre-test scores were reasonably close indicating comparative starting achievement. Multivariate tests revealed a significant difference (0.05) between the classes with respect to improved performance from pre to post when pre total score was used as a covariate to control for different levels of pre performance across the schools. Pairwise comparisons revealed that Andrea's Year 5 students demonstrated greater improvement in learning than the comparison Year 5 students from pre to post overall and on Part A and Part B of the standardised test.

DISCUSSION AND CONCLUSIONS

The improvement in Andrea's class from pre- to post-tests was remarkable, more than doubling the class average. This was much higher (over double the increase) than all but one other class (also at this school) of 40 classes in the larger project (Baturo, Warren & Cooper, 2003). This result is strong evidence of the following. First, improving teachers' mathematics knowledge is a powerful method to enhance students' mathematics learning outcomes if the teachers have strong pedagogy (RAND, 2000). Second, the improvement in teachers' mathematics knowledge has to be in understanding of structural knowledge (Baturo, 1998; Sfard, 1991), the big ideas of mathematics, and in the generic teaching processes such as reversing and nonprototypic representations (Hershkowitz, 1989; Krutetskii, 1976; RAND, 2000), the big ideas of mathematics pedagogy. Third, the cognitive difficulty of topics such as the part/whole fraction subconstruct are points at which students, even with good teachers, can fail, but this can be remedied by the techniques in this paper. Fourth, the importance of Entry and a good assessment instrument cannot be overlooked – they provide the foundation for teaching as well as assessment. Last, the story of Andrea shows that good teachers are not necessarily those who know, but are those who can

recognise and utilise powerful ideas when they see them. The support they need is the inclusion of these ideas in their daily teaching.

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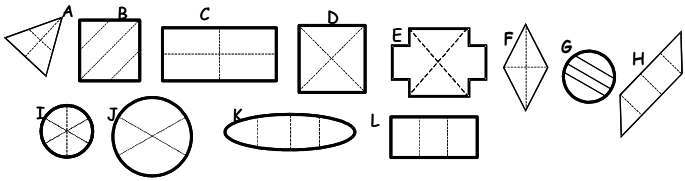
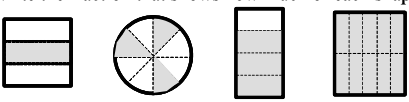
APPENDIX

C0GNITIVE DIAGNOSTIC COMMON FRACTIONS TEST

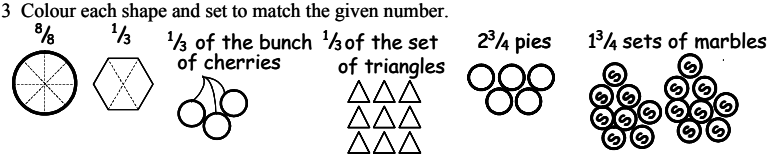
1 Tick the shapes below that have been divided into quarters.



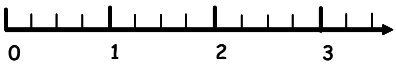
2 Write the fraction that shows how much of each shape has been shaded.



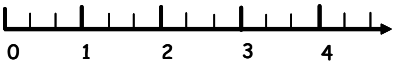
3 Colour each shape and set to match the given number.



4 Show $2^1/_4$ on the number line.

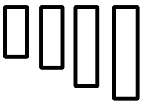


5 Write A to show where $^6/_3$ is on the number line below and write B to show where $^{11}/_6$ is.



6  This is $^1/_4$ of a ribbon. Draw the whole ribbon.

7  This is $^2/_3$ of one of the rectangles below. Tick the correct rectangle.



8

This is $^3/_5$ of a set of marbles.	Draw the set of marbles.
	

UNDERSTANDING INVERSE FUNCTIONS: THE RELATIONSHIP BETWEEN TEACHING PRACTICE AND STUDENT LEARNING

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This study is a part of an ongoing research that attempts to explain the relationship between the teachers' instructional practices and students' learning in the context of functions. In this paper we report a case that shows significant differences between the achievements of two classes irrespective of the students' background training, the curricula taught, and the geographic or socioeconomic variables. Cross examination of the data suggest that these differences are attributable to the teachers' instructional practices.

Introduction

The influence of the teachers' instructional practices on students' learning has prompted considerable interest (see for, example, Brophy & Good, 1986; Leinhardt & Smith, 1985). Directing this interest is the belief that teachers play an active and direct role in the students' acquisition of knowledge. During the 1970s teacher's effectiveness was measured in a quantitative way through the analysis of data associated with the courses taken by the teachers during their undergraduate studies or with teachers' scores on standard tests (Fennema & Franke, 1992; Wilson, Shulman & Richert, 1987). Such an approach is often criticised and found deficient because it is not associated with the situation where the teaching and learning take place.

More recently there has been a tendency to use qualitative research to investigate teacher efficiency in producing desired learning outcomes (Leinhardt & Smith, 1985; Askew, Brown, Rhodes, William, & Johnson, 1996). Leinhardt & Smith reported that expert teachers who had deep understanding of the concept of fraction obtained better learning results with their classes than did novice teachers. Teaching approaches of the latter was characterised by the provision of procedural examples and explanations but an absence of explicit links between different aspects of the concept. Askew *et* al concluded that the students of teachers who provided conceptual explanations and identified links between the sub-concepts (connectionists) obtained relatively better learning results in comparison to those students whose teachers encouraged them discover mathematical ideas and principles by themselves or those who were the recipients of dispensed knowledge. This paper takes the interest further by examining the way in which two Turkish teachers introduce the concept of inverse function and relates this to the students' understanding of the notion.

Theoretical Framework

Our study is situated, in general, in the process-product paradigm. To examine the teachers' instructional practices we draw upon Shulman's (1986) notion of pedagogic content knowledge "*the ways of representing and formulating the subject that makes*

it comprehensible to others" (p: 9). He suggests that such knowledge also includes the teachers' understanding of what makes the learning of certain topic easy or difficult for students, an understanding of the conceptions and preconceptions that students bring with them to the lessons and an awareness of students' misconceptions. We explain students' learning with reference to the APOS theory hypothesised by Dubinsky (1991) although we use only the first two aspects of this notion since we will show that the students did not appear to proceed to an object conception of inverse function. Dubinsky's notion of action refers to the repeatable mental or physical manipulations implemented upon an object to obtain a new one (Cottrill, Dubinsky, Nichols, Schwingendorf, Thomas, & Vidakovic, 1996). In our context those students whose understanding is limited to the action conception would work out the rule of inverse function by inverting the process of a function step by step. A process conception of a mathematical idea is attained through interiorising actions, and this level of understanding enables students to have a conceptual control over a process without necessarily performing every step in that process (Breidenbach, Dubinsky, Hawks, & Nichols, 1992). In our case, those who attained a process conception are likely to deal with the concept of inverse function in the situations that do not involve an operational formula.

The Notion of Inverse Function—The Turkish Context

The Turkish mathematics curricula within which our study is situated presents the concept of inverse function through a definition: *"Consider that f and g are two functions. If* $(fog)(x)=I(x)\Rightarrow f$ *is the inverse function of g and g is the inverse function of f",* and symbolises this relation as $f^{-1}(x)=g(x)$ and $g^{-1}(x)=f(x)$ (Cetiner, Yildiz, & Kavcar, 2000). This definition involves the idea that 'an inverse function undoes what a function does'. In this sense, the notion of 'undoing' captures the underlying domain of inverse function (Even, 1991). The property of 'one-to-one and onto' is the basic criterion that a function must meet to be reversed. What makes this cognitively simple mathematical idea difficult for many of the students is the peculiarity of the representations. Whereas Venn diagrams, sets of ordered pairs, and Cartesian graphs are more able to elucidate the essence of this concept, the absence of an algebraic formula in such situations usually creates difficulties for the learners unless they have attained a process conception (Dubinsky & Harel, 1992). We believe that algebraic expressions are likely to shift the focus of attention from the notion of 'undoing' to the idea of an 'inverse operation' entailing the inversion of a sequence of algorithms in the process of a function by going from the end to the beginning.

Method

This study was conducted in Turkey. The research participants were two high school teachers, Ahmet with 25 years teaching experience and Mehmet with 24 years teaching experience (the names are altered), and their 9th grade students. Data about the teaching practices were obtained through classroom observations. Each teacher was observed teaching the concept of inverse functions. All the lessons were audio taped and field notes were taken to record the critical information as well as the visual aspects of the lesson that the audiotape could not detect. Data about the students learning comes from two sources: pre-test and post-test questionnaires.

Preceding the courses a pre-test questionnaire was administered to the students to assess their initial levels of understanding of function, in general, and inverse function, in particular. After completion of the course a post-test was conducted to observe the progress students had made as result of the instructional treatment. The questions presented in this paper were used in the questionnaires in an open-ended form to encourage the students to write down their actual reasoning about the problems at hand.

Results

The results are presented in two ways. First we consider the overall approaches of the two teachers in teaching the concept of inverse function, and secondly we consider the responses of students from each of their classes (Ahmet Class A and Mehmet Class B) to two questions that focus on the notion of the inverse function.

The two teachers display substantial difference in their approaches to the essence of the concept, and this manifests itself in every aspect of their instructional discourse. Ahmet's teaching is centred on the notion of 'undoing'. In this respect, his first and purposeful attempt is to strengthen the students' understanding of 'one-to-one and onto' condition before the formal instruction. Diversity as well as development in the use of representations that started with Venn diagrams and went through a sequence that included the use of sets of ordered pairs, graphs, and algebraic expressions, were indicators of his expertise and essential to his determination to align the logic of the concept to the students' comprehension. Connections between ideas as well as between representations were a distinctive feature of his instruction. Ahmet's teaching was exemplified by his tendency to encourage his students to examine the concept through conceptually focused and cognitively challenging tasks. He believes that algebraic expressions, especially linear ones, are not productive to explicate the essence of an inverse function.

In contrast, Mehmet's teaching could be described as action oriented practices. He focused on teaching algorithmic skills and the acquisition of procedural rules. As his teaching developed, it became clear that these rules and skills were regarded by him as essential in enabling his students to reverse an algebraic function. However, such skills didn't help them to meaningfully deal with the concept in various situations. He made use of the students' previous knowledge and offered several analogies from daily life situations to encourage the students' acquisition of these procedural skills. Cartesian graphs and sets of ordered pairs were absent in his teaching. The ultimate goal of his instruction appears to be the alignment of the logic of 'inverse operation' to the procedural knowledge of 'doing' ("Find the inverse of…"), but not the conceptual knowledge of 'undoing'. To reach this target he worked on ritual tasks and consistently provided procedural explanations through the implementation of a 'focused questioning teaching strategy'.

From full analysis of the data we summarise the critical aspects of the teachers' instructional practices in the table below.

Ahmet	Mehmet
Preliminary Consideration Prepared students for the concept of inverse inverse function before formal introduction	
Introduction Explained the necessity of 'one-to-one' and 'onto' condition with reference to the definition of the function and through several examples in the form of Venn diagrams…	Provided several analogies from the daily life situation to explain the way of inverting a sequence of operations in the process of the function…
Development Examined the concept of inverse function a through the Venn diagrams, sets of ordered pairs, graphs, and algebraic expressions…	Concept examined through Venn diagrams and algebraic expressions. Sets of ordered pairs and graphs ignored. With reference to the definition used a single example in the form of Venn diagram to explain the necessity of 'one-to-one and onto' condition.
Expansion Making use of the students' knowledge of 'inverse operation' when teaching linear functions in algebraic forms… Attempted to expand the students' understanding of inverse function as 'undoing' what a function does through conceptually focused and cognitively challenging tasks…	Did not engage students with conceptually focused and cognitively challenging tasks… Largely confined the notion of inverse function to the idea of 'inverse operation'…
Pedagogical Characteristics Displaying a mixed approach (connectionist & discovery) as a teaching strategy…	Implementing a focused questioning method as a teaching strategy…

Table 1: Salient aspects observed in teachers' instructional practices.

Prior to the course all of the students were asked to demonstrate their ability to reverse a process after being given a particular output (5) after completing the processes x3, –7. Only one student gave an incorrect solution. Solution methods of the students who obtained correct answers were almost equally distributed between the formation of an algebraic equation or an inverse operation. Differences in the students' understanding after the course may be seen through the analysis of two questions. The first assesses students' understanding of the notion of 'undoing' and the property of 'one-to-one and onto' whilst the second investigates their ability to deal with the concept of inverse function in a graphical situation.

The First question asked the students to*:*

Consider two non-empty sets, A = {a, b, c, d} and B = {e, f, g}. Is it possible to define a function from A to B, say f, that has an inverse function, say f^{1}? Give your answer with the underlying reasons.

Within this question there is neither an explicit recipe nor a visual figure to facilitate the students' movement between the sets of elements. They had no choice other than to construct a process in the situation without losing the meaning of inverse function and the related properties. Five different responses were produced (see table 2).

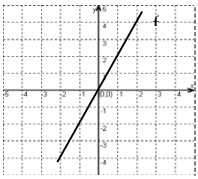
	Class A		Class B	
	n	%	n	%
Incorrect (verbal explanation)	5	18	1	4
Incorrect (verbal explanation & corresponding figure)	1	4	9	33
No response	3	11	3	11
Correct (verbal explanation)	10	36	5	19
Correct (verbal explanation & corresponding figure)	9	32	9	33
Total (N)	28		27	

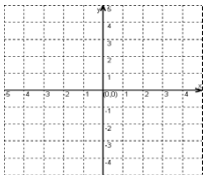
Table 2: Distribution of the answers by methods used and correctness.

Incorrect verbal explanations did not make sense or articulated an idea that illustrated a misunderstanding the concept of inverse function — *"...we cannot define such a function, because the sets A and B do not have a common element."* The common error in the second type of answers is about the univalence condition. Although students who made this error flexibly shifted to visual figures, mainly Venn diagrams, they either constructed a 'one-to-one' relation from A to B and then claimed that it has an inverse function, or defined a proper function from A to B ignoring the univalence condition on the way back. One third of students in Class B (Mehmet's class) provided incorrect explanations though they worked on a visual figure. Only one in Class A (Ahmet's class) did so. Approximately one quarter of the total number of students appear to have a cognitive control over the processes in both ways. These students explained verbally why the construction of such a function is not possible with a clear articulation that *'an inverse function undoes what a function does'* with a particular emphasis upon *'one-to-one and onto'* condition. They did not use a visual figure to justify their thoughts. However, again class differences appear. For each student who displays this characteristic in Class B there are two students in Class A. The last group of answers also indicates the recognition of what an inverse function does and the property of 'one-to-one and onto'. However, though it is difficult to make a decision about the mode of students' thinking on the basis of written responses, it is inferred, from the evidence presented, that these students were dependent upon a visual figure to think about the problem.

The second question that we will consider was presented in graphical form.

The graph of function f is given as follows. Sketch the graph of inverse function, f^{-1}, in the Cartesian space below, and give the reasons for your answers.





Excluding those who gave no response this question produced three types of answers (see table 3)

	Class A		Class B	
	n	%	n	%
Incorrect	7	25	15	57
No response	0	0	2	7
Correct (point-wise approach)	14	50	10	37
Correct (global approach)	7	25	0	0
Total (N)	28		27	

Table 3: Distribution of answers by methods used and correctness.

Incorrect responses involved several types of misunderstandings, such as sketching a line passing through the points (2, 0) and (0, 1) on the x and y-axes respectively, sketching the graph given as the graph of an inverse or reflecting the graph of the function given in the y-axis. Note that almost two thirds of class B gave an incorrect response or no response. Correct responses involved two qualitatively different approaches. The first group of students displayed a point-wise approach either by marking certain points, such as (2, 1), (4, 2), (-2, -1), in the Cartesian space and then drawing a straight line through them or using the algebraic form of the function for transition from the graph given to that required. The second group of students, all of whom are in class A, sketched the graph of inverse function at once without any attempt to deal with the graph point by point. The common method is reflecting the graph given in the line of $y = x$.

Conclusion

The impact of teaching practices on students' learning is a fruitful but at the same time a controversial research topic. Whereas educational sociologists emphasise the complexity of the social environment, within which there are several other variables that would profoundly affect the students' learning (Peaker, 1971), educational psychologists argue that the individual's cognitive growth is the most determinant factor in his/her acquisition of knowledge (Inhelder & Sinclair, 1969). We are fully aware that the impossibility of eliminating all the internal and external factors does not allow us to explain the influence of teaching practices on students' learning in the sense of cause-and-effect relationships. However, our findings suggest that teaching practices that differ in a qualitative way are apt to produce qualitatively different learning outcomes. The epistemology of the inverse function was the basic criterion

in our examination of the students' learning, the teacher's teaching practices, and the interaction between the two. We conclude, primarily, that students would have difficulty in attaining a meaningful understanding of inverse function without experiencing it through conceptually focused and cognitively challenging tasks using a variety of representations. Making use of students' previous knowledge (the knowledge of inverse operation) or providing analogies from real life situation might be productive for the construction of a foundation, but it is not adequate enough to promote the students' conceptual understanding of inverse function. We suggest that what determines the quality of teaching, and would subsequently enhance the students' meaningful learning, is making use of a variety of appropriate representational systems, examining the concept through conceptually focused and cognitively challenging tasks, linking the inverse function to the concept of 'one-to-one and onto' function as well as to the concept of function itself, and ensuring active involvement of the students within the process of knowledge construction.

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THE IMPACT OF TEACHERS' PERCEPTIONS OF STUDENT CHARACTERISTICS ON THE ENACTMENT OF THEIR BELIEFS

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This paper reports on one aspect of a larger study and comprises an analysis of the beliefs concerning mathematics, its teaching and its learning, and the classroom practice of one secondary mathematics teacher. It focuses on the question, "What specific teacher beliefs about students are relevant to teachers' classroom practice in various classroom contexts?" The teacher's practice was examined in relation to several of his mathematics classes and significant differences, consistent with the teacher's beliefs in regard to the various classes, were found. The findings confirm the contextual nature of beliefs and highlight the importance to teachers' practice of specific teacher beliefs about the various students that they teach.

BACKGROUND AND THEORETICAL FRAMEWORK

A fundamental premise of teacher beliefs research has been that an individual's behaviour is ultimately a product of his/her beliefs (Ajzen & Fishbein, 1980; Cooney, 2001). Consequently, any attempt to change the practice of teachers must, of necessity, involve change in the beliefs of teachers. Teachers' beliefs have, therefore, long been regarded as critical to the reform of mathematics education (Cooney & Shealy, 1997). Despite this there is no agreed definition of the concept of beliefs (McLeod & McLeod, 2002). It is thus the responsibility of researchers in the area to make clear the meaning that they attach to the term (Pajares, 1992). In this paper "beliefs" is used to mean anything that a person regards as true, and is essentially the meaning assigned to the word by Ajzen and Fishbein (1980). Furthermore, since beliefs must necessarily be inferred (Pajares, 1992), more certainty can be attached to the existence of a belief that is evident in both the words and the actions of an individual. Indeed, the degree to which a subject's actions and statements in other contexts are compatible with a given stated belief, the more centrally held (Green, 1971) that belief is likely to be. It is also recognised that individuals may hold beliefs that they do not articulate for a variety of reasons, including the fact that they may not be consciously aware of their existence (Buzeika, 1996).

Wilson and Cooney (2002) observed that since the 1980s context has been increasingly recognized as relevant to studies of teaching and learning and that the teacher's beliefs in fact constitute part of the context in which classroom activity occurs. In their theory of planned behaviour, Ajzen and Fishbein (1980) emphasised the context specificity of beliefs and Green (1971) also asserted the relevance of

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context to the enactment of beliefs, suggesting that the relative strength with which various beliefs are held is dependent upon the particular context.

Contextual constraints have also been recognised as exerting significant influence on the relationship between beliefs and practice (Sullivan & Mousley, 2001) while Hoyles (1992) described all beliefs as situated as a consequence of their being constructed as a result of experiences which necessarily occur in contexts. Hoyles (1992) argued that it is thus meaningless to distinguish between espoused and enacted beliefs or to examine the transfer of beliefs between contexts since differing contexts will, by definition, elicit different beliefs. Thus, rather than contextual factors constraining teachers from implementing certain of their beliefs, such factors in fact give rise to different sets of beliefs which are indeed enacted. Such a view is consistent with that of Ajzen and Fishbein (1980). Pajares (1992) also stressed the contextual nature of beliefs and the implications of their being held, not as isolated entities, but as part of belief systems as described by Green (1971).

Context is thus relevant to both the development and the enactment of teachers' beliefs, as well as to the particular beliefs that are relevant in a given situation. Hence an important challenge for researchers is to identify specific teacher beliefs that significantly impact their practice and that, while context specific, are relevant across a sufficiently broad range of contexts to be generally applicable. Hoyles (1992) described the emergence within PME of research that contributes to this end. In particular, she cited Romberg (1984) as identifying a relationship between teachers' beliefs about students' ability and the nature and difficulty of the tasks that they assign to them. Hoyles (1992) also called for more attention to be paid to the study of teachers' beliefs as they exist in relation to various specific contexts and particularly in relation to the characteristics of their students.

In spite of this there is still little knowledge regarding specific teacher beliefs in relation to students that are likely to be helpful or otherwise in the creation of classrooms that reflect the principles of mathematics education reform. Exceptions include the finding of Stipek, Givvin, Salmon and MacGyvers (2001) that teachers who believe that students' mathematical ability is fixed are more likely to hold traditional views of mathematics teaching, and Cooney, Shealy and Arvold's (1998) findings regarding the beliefs of pre-service secondary mathematics teachers. While acknowledging the context specificity of beliefs they identified a number of beliefs that such teachers tend to hold about themselves and their role as a teacher and also made use of aspects of Green's (1971) description of belief systems in accounting for both the varying impacts of these beliefs on teachers' practice and their susceptibility to change. This paper reports on an examination of the variations between classes in the beliefs and practice of an individual experienced mathematics teacher, enabling insight into the nature and place within the structure of the teacher's beliefs system, of his beliefs with respect to students.

Practice, in the current study, was considered in terms of the extent to which the teacher's classroom environments could be characterized as constructivist. This was done cognisant of the facts that constructivism is not prescriptive in relation to teaching (Simon, 2000) and that any teaching strategy could be part of a constructivist learning environment (Pirie & Kieren, 1992). Rather a constructivist classroom environment was considered to be one in which: students were able to act autonomously with respect to their own learning; the linking of new knowledge with existing knowledge was encouraged and facilitated; knowledge was negotiated by participants in the learning environment; and the classroom was student centred in that students have opportunities to devise and explore problems that are of relevance to them personally (Taylor, Fraser & Fisher, 1993). Such elements align well with the principles and standards promoted by the National Council of Teachers of Mathematics (NCTM) (2000).

THE STUDY

The subject

Andrew had been teaching secondary mathematics and science for 25 years. He had studied mathematics for three years at University as part of his B.Sc. and had since completed an M.Ed. Andrew was currently teaching mathematics to two classes in grade seven and one in grade ten. Both of the grade seven classes were heterogeneous while the majority of students in the grade ten class were in the average ability stream with a few studying a separate mathematics course designed for low ability students.

Instruments

Data concerning Andrew's beliefs were collected using a survey requiring responses on a five-point Likert scale, to twenty six items relating to beliefs about mathematics, its teaching and its learning, and from a semi-structured interview of approximately one hour's duration. The survey items were taken from similar instruments devised by Howard, Perry, and Lindsay (1997) and Van Zoest, Jones, and Thornton (1994) and were originally part of a forty-item survey that was shortened after use in a pilot study. The audio-taped interview required Andrew to: reflect upon his own experiences of learning mathematics; describe an ideal mathematics classroom and compare this with the reality of his own mathematics classes; respond to 12 statements about the nature of mathematics based upon the findings of Thompson's (1984) case studies of secondary mathematics teachers; and respond to a further 12 statements about the teaching and learning of mathematics derived from the same source. The 12 statements regarding the nature of mathematics consisted of four each that represented Problem Solving, Platonic, and Instrumentalist views of mathematics as defined by Ernest (1989), and the 12 statements relating to the teaching and learning of mathematics were similarly representative of three corresponding views of mathematics teaching and learning.

Observations of approximately six lessons with the classes in each grade provided data on Andrew's classroom practice as well as further opportunities to gather data from which his beliefs could be inferred. Data on Andrew's classroom practice were also gathered from the interview and from both teacher and student versions of Constructivist Learning Environment Survey (CLES) described by Taylor et al. (1993) and requiring respondents to indicate on a five-point Likert scale, their perceptions of the frequency of various teaching/learning practices in their mathematics classroom. The CLES measures the four aspects of a classroom environment described above and respectively named Autonomy, Prior Knowledge, Negotiation and Student-Centredness (Taylor et al., 1993).

Procedure

Andrew completed the beliefs survey during the first few weeks of the school year. After a gap of several weeks he was asked to complete the teacher version of the CLES with respect to at least of two of his mathematics classes (one grade seven and the grade ten), and then to give the student version of the survey to students in these classes. Interviews were conducted in early October and observations of Andrew's mathematics lessons occurred throughout November and December. Inferences concerning Andrew's beliefs were made on the basis of the complete data set. That is, Andrew's interview transcript and the detailed notes made during and after each observation period were examined for evidence supporting, contradicting or clarifying his belief survey responses and the CLES responses of both Andrew and his students. A set of five centrally held beliefs that emerged as most relevant to Andrew's practice were suggested and put to him, along with details of the data analysis, for comment and verification.

Results and discussion

Andrew's belief survey responses indicated that he held a Problem Solving view of mathematics (Ernest, 1989) and a constructivist view of mathematics learning. This was exemplified by his agreement or strong agreement with statements such as the following:

> Mathematics is a beautiful, creative and useful human endeavour that is both a way of knowing and a way of thinking.
>
> Ignoring the mathematical ideas that children generate themselves can seriously limit their learning.
>
> A vital task of the teacher is motivating children to resolve their own mathematical problems.

However, Andrew seemed unsure as to the most effective pedagogical approach to employ in enacting those beliefs. For example, he was undecided about the following items:

> Mathematical material is best presented in an expository style: demonstrating, explaining and describing concepts and skills.

> Providing children with interesting problems to investigate in small groups is an effective way to teach mathematics.

Students in one of Andrew's grade seven classes and his grade ten class completed student versions of the CLES. Andrew opted to complete just one CLES (teacher version) survey rather than one for each class. The three sets of responses were similar for three of the four scales with the exception being the extent to which the classroom environments were perceived to be Student-centred. Both classes perceived their classrooms to be less Student-centred than did Andrew, with the difference being greatest in the case of the grade tens. Individual items that contributed to these differences are shown in Table 1. While individual differences are small, the data suggest that in his grade ten class Andrew was more likely than in his grade seven class to set the tasks and to be the arbiter of correct solutions.

Table 1: Items contributing to differences in Student-Centredness

In this class…	Teacher	Grade seven (av. response)	Grade ten (av. response)
I/the teacher give the students problems to investigate	Seldom	Sometimes	Often
The activities students do are set by me/the teacher	Often	Often	Very often
Students learn my/the teacher's method for doing investigations	Sometimes	Often	Often
I/the teacher show(s) the correct method for solving problems	Often	Often	Very Often

Andrew's interview responses confirmed his Problem Solving view of mathematics and his constructivist view of learning as conveyed in his belief survey responses. The following quotations are illustrative:

> I've swung in the past from being a Platonist to being more a social constructivist. So mathematics is, I don't think it's out there, I don't think there is a number one sitting somewhere in the ether however effectively it sits there because we as a society have created it.

> Mathematics to me is for exploring, conjecturing…, well, there's probably no such thing as right answers to any problem.

His ambivalence regarding teaching approaches was also evident in that he expressed agreement that one of the ways that students learn is "by attentively watching the teacher demonstrate procedures and methods for performing mathematical tasks, and by practising those procedures", but was careful to stress that this was just one way

that they learn, and that "there's got to be a big balance". Andrew described his own teaching as follows:

> I suppose I'm very teacher directed but at the same time what I like to do is not to give the kids the answers, but what I try to do is to make them think …

Andrew talked primarily about his grade seven classes and described his grade tens as "a totally different kettle of fish". He described them as more difficult to motivate, and also talked about what he regarded as the inappropriateness of much of the content of the course designed for grade ten students in the middle ability level, suggesting that these students needed "survival numeracy skills" and not "space and all these other things". In the lessons observed, the grade ten class worked from worksheets on coordinates and mapping. Apart from brief introductions there was no whole class teaching. Rather students worked with minimal noise either individually or in twos or threes as they chose. Andrew rarely directly answered students' questions about the mathematics, but instead made suggestions. Several of the students did very little work, and this was largely ignored provided that their behaviour was not disruptive.

In contrast with his grade ten, Andrew's grade seven lessons consisted primarily of whole class discussions facilitated and guided by Andrew. Both of the grade seven classes seemed accustomed to being asked to explain their answers and comfortable with going to the board to write their answers. Questions such as, "How did you do it?", "Is it right?" and "Will it always work?" were recurrent with the students sometimes using them too. The students consistently appeared to be engaged in genuinely grappling with the meaning of the mathematics. In one lesson on multiplying fractions, Andrew allowed the students to run with a conversation without comment until a student articulated that the effect of multiplying by a number greater than one makes it bigger and multiplying by a number smaller than one makes it smaller. Such conversations remained orderly, with one person speaking at a time and everyone else listening.

Andrew's beliefs and practice

Three beliefs emerged as centrally held and relevant to Andrew's teaching in both the grade seven and grade ten classes. These were:

1. The teacher has a responsibility to maintain ultimate control of the classroom discourse.
2. The teacher has a responsibility actively to facilitate and guide students' construction of mathematical knowledge.
3. The teacher has a responsibility to induct students into widely accepted ways of thinking and communicating in mathematics.

However, although Andrew's teaching in both contexts was consistent with constructivist principles there were clear differences. Andrew agreed that he was less

inclined than with the grade seven classes to make the effort to maximize the engagement of all students in the grade ten class, or to establish the social norms required to make whole class teaching effective for all students. Two further centrally held beliefs underlay this, namely:

4. Older students of average ability are not interested in mathematics.

and,

5. Mathematics that is suitable for older students of average ability is not interesting.

CONCLUSION

While the teacher in this study held beliefs that were essentially consistent with the aims of the mathematics education reform movement it is clear that his beliefs in relation to older students of average ability had a significant impact on his practice in their lessons and in fact limited the extent to which at least some students this class were likely to engage in mathematical thinking as embodied in the NCTM's (2000) process strands. Thus if our aim is to promote teaching that is consistent with a constructivist view of learning then it is insufficient to assist teachers to develop beliefs that are considered helpful to this end without attending to other beliefs that they may hold in relation to specific contexts.

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TOWARDS THE EMERGENCE OF CONSTRUCTING MATHEMATICAL MEANINGS

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This paper presents the design and some results of a series of teaching experiments. The design was created to develop a model for everyday maths lessons, that describes the conditions which foster or hinder the construction of new mathematical meanings. The development process includes the students' epistemic processes, their social interactions, the mathematical domain and supporting functions of the teacher.

INTRODUCTION

The theory of interest-dense situations describes how mathematical meanings are constructed in a special kind of situations in everyday maths lessons, so called interest-dense situations. (Bikner-Ahsbahs 2002, 2003a, 2003b) At the same time, these situations characterize favourable conditions which support interest in mathematics. Thus, fostering interest in mathematics and constructing new mathematical meanings mesh together in interest-dense situations. However, new mathematical meanings are not only constructed in interest-dense situations but also during situations which cannot be called interest supporting at all. How can the condition network out of which constructions of new mathematical meanings emerge theoretically be grasped independently whether interest is supported or not?

Based on empirical data the three collective epistemic actions gathering and connecting meanings and structure-seeing were reconstructed during the development of the theory of interest-dense situations. These actions accompanied all processes of constructing new meanings during interest-dense situations and were the basis for building ideal types of epistemic processes. (Bikner-Ahsbahs 2003a, 2003c) Gathering and connecting meanings are activities which provide the tools for students to see mathematical structures whose validity they prove afterwards. The interplay of gathering and connecting meanings on the one hand, and situation directed student and teacher behaviour (see also Williams 2002, 2003), on the other hand, could be regarded as a basic condition for the emergence of new mathematical meanings within interest-dense situations in the sample class (Bikner-Ahsbahs 2003a, 2003b). The question now is whether this condition network can be transferred to other classes, students of another age and other mathematical domains.

Following this question I designed a series of teaching experiments in order to investigate the conditions which support or hinder constructing new mathematical meanings about infinite sets. In the long run my goal is to develop a theory modelling the emergence of mathematical meanings in everyday maths classes.

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THEORETICAL BACKGROUND

During the last few years Dreyfus, Hershkowitz and Schwarz have developed a theory to analyse processes of abstractions. (Dreyfus/Hershkowitz/Schwarz 2001; Hershkowitz/Schwarz/Dreyfus 2001) Meanwhile this theory has been adopted in different empirical studies. (Stehlikova 2003; Tsamir/Dreyfus 2002; Tabach/Hershkowitz 2002; Williams 2002, 2003) The authors regard abstraction as a cultural activity leading to the construction of new meanings while reoganizing and restructuring familiar mathematical knowledge into a new structure. Processes of abstraction are driven by needs or motives. (Hershkowitz/Schwarz/Dreyfus 2001). The core concept of this theory is an epistemic action model connecting three epistemic actions: *recognizing*, *building-with* and *constructing*. *Recognizing* refers to recognizing a familiar structure. *Building-with* is seen in the process of combining familiar pieces of knowledge into a new context. It includes recognizing. Processes of restructuring and reorganizing what is recognized, and known to construct new meanings are labelled *constructing*. The authors call this epistemic action model the dynamically nested RBC-model because of the nested and dynamically interwoven characteristic of the epistemic actions.

This model describes an inner-perspective of constructing new meanings dependent on the situational conditions, the biographical background and the interactional possibilities. Whether or not utterances indicate recognizing, building-with or constructing is due to the students' biography and their individual abilities. Since the development of this theory was a process of abstraction itself the same should be true for the RBC-model. The authors claim the RBC-model to be a suitable instrument for analyses of learning processes that fit the definition. Indeed, available data from which the authors extracted their theory only show teaching experiments with didactical designs for the construction of new knowledge that step by step and systematically builds upon previously constructed structures. Thus, the RBC-model is especially useful for analyses of the just described construction processes. To what extent it is suitable for the analyses of more open situations with spontaneous constructions of meanings is an open question at present.

The model of collective epistemic actions, gathering and connecting meanings and structure-seeing has been extracted from data describing interest-dense, hence, more open situations with a wide range of opportunities for the student to spontaneously construct new meanings. This model characterizes an outer-perspective because it is not really necessary to know details about the students' learning biographies in order to decide what kind of action occurs. Weather or not a structure is new, the group usually indicates during the interactions. To what extent both models can be combined into an integrative model that characterizes the emergence and the process of constructing new mathematical meanings is the key question in this project.

Nearly all data that document processes of the emergence of new knowledge show: If teachers participate in these situations they mould them to a certain extent. Therefore an integrative model of epistemic actions should include the supportive function of

the teacher. Although teachers are included in investigations with both models concerning social interactions they are not integrated into the models directly. This has yet to be done with an integrative model of epistemic actions.

METHODICAL AND METHODOLOGICAL REFLECTIONS

Comparing infinite sets as an initial activity

Tsamir and Dreyfus have reconstructed the process of constructing meanings of Ben, a talented student of grade ten. During the interview they provoke different representations of countable infinite sets of numbers and evoke a contradiction about their size comparing the natural numbers. This contradiction induces Ben to reflect on the contradiction itself. This way he constructs meanings on different levels: on the mathematical content level and on a more reflective level of building mathematical theories (Tsamir/Dreyfus 2002). The central activity in these teaching experiments is comparing sizes of infinite sets. Results from this case study may be integrated into the development process of the integrative model if the design of gathering data is created in a similar way to that of the two authors.

I have taken up the basic idea of comparing infinite sets. This idea is included in the creation of a series of experiments. The probands are students of grade nine. The task in the teaching experiment consists of a preparation and core task.

Preparation task: One card after another is uncovered from a pile of cards. The cards show the natural numbers as a sequence to 7. Then cards from another pile are uncovered. This time the cards show the squares of the natural numbers to 49 (figure 1). The rest of the two piles are put beside the two sequences.

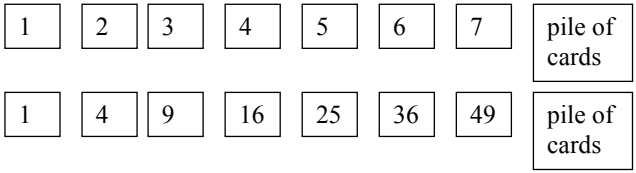

Figure 1: Comparing infinite sets of numbers using stacks of cards

Imagine you could write all natural numbers and all square numbers on cards. Which pile of cards would need more cards? Why? For assistance further piles are prepared. Every pile shows different numbers for each pair of cards taken. Each pile is presented one by one but only if it seems necessary.

Core task: The next representation schema made up of fractions is presented and explained (figure 2). The first row consists of all fractions with the denominator 1, the second one with denominator 2, and so on. Are there as many natural numbers as there are fractions? Why?

$$
\begin{array}{ccccc}
\cdots & \cdots & \cdots & \cdots & \\
\frac{1}{4} & \frac{2}{4} & \frac{3}{4} & \frac{4}{4} & \cdots \\
\frac{1}{3} & \frac{2}{3} & \frac{3}{3} & \frac{4}{2} & \cdots \\
\frac{1}{2} & \frac{2}{2} & \frac{3}{2} & \frac{4}{2} & \cdots \\
\frac{1}{1} & \frac{2}{1} & \frac{3}{1} & \frac{4}{1} & \cdots
\end{array}
$$

Figure 2: Representation schema for all fractions

The series of experiments – an overview

The development of insight during this research process is conceptualized as an iterative process. The series of experiments consists of teaching experiments. It starts with laboratory experiments and it finishes with field experiments. Each step consists of the conduction of the experiment, gathering and analysing all the data of the step at the current experimental stage. After extracting hypotheses from the analyses the next experiment is prepared and carried out. Using the concept of theoretical sampling from the grounded theory (Strauss 1994) sampling is theoretically based on previous insights in order to prove hypotheses and to continue the theoretical trace carried out before. The design is carefully adapted according to the choice of probands and a theory based variation of the didactical and interactional conditions.

To get a deeper insight into different theoretical perspectives analyses are adapted to the principle of triangulation. Data from each experiment are analysed from three different theoretical perspectives:

- From an epistemological perspective
- From a situational perspective
- From an individual perspective

In the first perspective the two epistemic action models are used, in the second and the third perspective the situational conditions are analysed using background theories of social interactions with a focus on the social environment, on the one hand, and the students' and the teacher's behaviour on the other.

The first experiment as an endurance test

The series of experiments start with so called endurance experiments. This means:

- The design will prove the capacity of the two epistemic models to some extent.
- The tasks are not separated into systematically assembled steps but are arranged in an open way to give the students as much leeway as possible for spontaneous and unexpected constructions of meanings.
- The interviewer acts in two roles, the role of a teacher and the role of a test administrator. She restricts herself as much as possible to the role of the researcher.

She presents material and ensures that the process of constructing meanings continues until the problems are solved. To shape a friendly atmosphere she tries to understand the students' utterances from their view. At the same time she avoids as far as is possible giving assistance concerning the content, emotional support or acknowledgement.

The teaching experiments begin with students who are interested in mathematics and voluntarily solve mathematical problems, for the probands should be able to cope with endurance situations and still reach the level of constructing new mathematical meanings. In the course of the series of experiments more and more, not necessarily interested students will be included in the sample.

Three kinds of data: informing – acting – commenting

The three-step-design of each experiment consists of three phases assembling different modes of data, which are included in the analysis of each iteration step (Busse/Borromeo Ferri 2003). The phases are video recorded. The first phase lasts about five or ten minutes, the second about an hour and the third has no time limit:

- Phase 1: The probands are asked what infinte means to them (informing).
- Phase 2: The probands work on the tasks (*acting*).
- Phase 3: The probands and the researcher watch the video records of the second phase. The students stop the film and comment on the situations at important points. They are asked to say how they experienced the process of working on the problems (c*ommenting*).

SOME DATA, RESULTS, AND CONCLUSIONS

The transcipt of the first experiment is worked out but the analyses are not yet finished. Nevertheless, they already show interesting tendencies. Both epistemic action models seem to complement each other. The model of collective epistemic actions describes collective gathering and connecting activities where familiar structures are recognized and which are the basis for building-with processes. If, like in our case, no mathematical knowledge about infinite sets is available the attempt to link the new context and previously constructed structure seems to be obvious. The students' metaphors link known mathematical structures as source systems with the new context to enlarge the range of possible actions with new objects.

Example 1: Toni says *infinite* is a number that is bigger than any other number. This idea is elaborated by Toni and his friend Robin in common: "this number exists on a theoretical basis only", "the number infinite (infinity) is a theoretical assumption which never can be proved", "probably we have a theoretical number that probably could exist from the theoretical view but that does not really come out anywhere in practice". Therefore infinite is conceived as a theoretical number but not as a practical one. The metaphor *infinite is a number* enlarges action possibilities and broadens the students' number concepts. Along with this and during the further going second phase infinite is used as a counting number with special rules: $\infty:3 = \infty$, $\infty - 2 = \infty$, $(\infty - 2):2 = \infty$, $(\infty - 2):4 = \infty$, ... More meta-

phors like *infinite as a process, infinite as one point infinitely far away, infinite as a term* enable the students to use further reaching considerations and new kinds of acting.

Thus, possible considerations of infinite are gathered. They are implicitly shown in the students' actions and in their usage of metaphors. Metaphors link familiar sign systems with the potentially new sign system. They provide a choice for recognizing familiar mathematical objects which are combined and worked out on a trial basis. If Robin and Toni reach contradictions or limits during activities of building-with they restart gathering and connecting. This way they gain new tools and thus, an extended basis for recognizing and building-with.

The following hypothesis is derived from the preparation task and can be already proven in the analyses of the core task and of some data from other investigations:

> *Rich and fruitful problem based gathering and connecting activities are basic activities for the emergence of recognizing and building-with actions.*

Recognizing and building-with activities do not always lead to constructions. I will refer to a scene now where the two boys succeed in constructing new meanings. Through this I will show how interaction processes contribute to the construction of new meanings and will use this insight to deduce an important support function of teacher.

Example 2: During the preparation task Robin and Toni compare the size of the set of natural numbers with the size of the set of natural numbers beginning with 3. In the remark "you may balance it out again and again" Robin uses a the metaphor of *balancing out something*. By this he seems to explain to himself that the sizes of the sets of natural numbers beginning with 1 and beginning with 3 are equal. However, Robin does not work his idea out, not even when the interviewer asks him to do so. Instead, Toni elaborates it:

> Toni: yeah you can if you just take these two number rows (points at 1,2,3,4, ... and ,then at 3,4,5,6, ...) let's assume the infinity goes on and on and it is now at ,ten then this is at ten too (points at 3,4,5,6, ...) but includes two less but then it goes on until twelve. This is balanced OUT again. If that one goes on until twelve (points at 1,2,3,4, …) then that has two less. (points at 3,4,5,…)

Although the boys had never used an inductive proof before, Toni's way of argumentation is based on the idea of it and the preconception of an infinite process. He focuses on the first ten steps of the infinite process of uncovering cards written with natural numbers whose last number is ten. The segment of the comparison set of natural numbers beginning with 3 and finishing with the number 10 has two numbers less. The idea of an infinite process allows the supplement of the next two numbers to 12: "that is balanced OUT again" Toni says. Now he focuses on the segment of natural numbers to 12 which has two numbers more than the comparison segment. Continuing his thought process Toni lengthens the segment of natural numbers beginning

with 3 to the number 14. Afterwards this thought process is worked out in more detail.

Robin has used – probably in an unconscious way – a metaphor which he is not able to work out. Obviously Toni recognizes its potential for reasoning and constructs a way of arguing that confirms the validity of the statement that both sets have the same size.

Example 3: A similar phenomenon can be observed during the core task. Different attempts to construct all fractions as a sequence have failed. The central obstacle seems to be the order type of the set of fractions: The set of positive fractions does not have a minimal element and is dense. All of a sudden Toni starts to count beginning with $\frac{1}{1}$ ("one oneth") while moving his fingers from one fraction to another in a diagonal counting pattern (figure 2). Despite the request of the interviewer Toni does not react. He does not seem to be aware of what he is doing. A bit later Robin takes up the way Toni has counted and shows that the set of natural numbers and the set of fractions are equivalent.

The *metaphor of balancing out something* as well as the counting finger motion is a subconsciously used sign. These signs transform the situations into situations with an increased range of action. Obviously more can be expressed, said or shown than consciousness may grasp explicitly. The metaphor of balancing out something and the counting finger motion appear as an *offer of work* for the other student who now has extended possibilities to act. Exactly this idea to transform the situation through *offers of work* which can extend action possibilities underlines the importance of the teacher's support function. Re-analyses of data about interest-dense situations confirm:

> *Teachers are able to transform situations constructively and these transformations of situations support the construction of new mathematical meanings even if students are not especially interested in mathematics.*

This is shown by the use of *metaphors* and *offers of work* like motions, diagrams modelling student behaviour, or patterns of sketch-program actions where the teacher takes up a student's action sketch and transforms it into an action program. But how can support functions be included into an integrated model for the emergence of new mathematical meanings? This is an open question at present that has yet to be investigated.

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IDENTITY, KNOWLEDGE AND DEPARTMENTAL PRACTICES: MATHEMATICS OF ENGINEERS AND MATHEMATICIANS

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This report explores first year undergraduate mechanical engineering and mathematics students' conceptions of the derivative and the contribution that membership of different departments may have on these conceptions. Quantitative results suggest that mechanical engineering students develop a proclivity for rate of change aspects of the derivative whilst mathematics students develop a proclivity for tangent-oriented aspects. The analysis of qualitative results further suggests that students' conceptual development of the derivative and their emerging identities are closely related and co-evolve in accordance with the departmental practices in which learning and teaching occur.

INTRODUCTION

This paper explores first year undergraduate mechanical engineering and mathematics students' differing orientations to, or conceptions of, the derivative. The mathematical education of engineers is a topic of increasing debate (see, Kent & Noss, 2002; McKenna, McMartin, Terada, Sirivedhin, & Agogino, 2001; Maull & Berry, 2000). Researchers have dealt with mathematics in the practice of engineers, developing new curricula for engineering students, students' difficulties with understanding mathematics and students' conceptual development in specific topics. Maull & Berry (2000) has commonalities with our study. They examine first and final year mechanical engineering and mathematics undergraduates alongside postgraduate students and professional engineers. They conclude that "the mathematical development of engineering students is different from that of mathematics students, particularly in the way in which they give engineering meaning to certain mathematical concepts" (ibid, p.916). They noted that both groups of students showed similar patterns of responses at the entry, by the final year, the groups' responses diverged. They did not, however, provide reasons for this emergent divergence and called for further research:

> *'There is evidence in the literature that engineering students are socialised into ways of thinking and behaving, and we may ask whether the difference found stems from socialisation, from the interactions between students and their peers, lecturers and other professional contacts, or whether there is also a second acculturation process through their discovery of what is useful in the context of their study and work' (ibid, p.916).*

Our considerations of divergence in engineers and mathematics students' conceptions have focused on issues of identity. Western research on identity has, by and large, come from a psychological perspective that conceptualises identity as an individual's sense of self (Harter, 1998). In the last five years, however, considerations of identity have been informed by social and socio-cultural theories (Wenger, 1998; Holland Lachicotte, Skinner, & Cain, 1998; Boaler, 2002; Nasir, 2002). Common to these

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perspectives is an assumption that individual growth and learning is situated in the social context/practice in which learners/practitioners live – the 'process of becoming' is practice-bound. We draw on Holland et al.'s (1998) idea of 'positional identity' in this paper. Positional identity refers to the way in which people figure out and enact their positions in the worlds in which they live. Their account of positional identity is based on their theory that identities develop in and through social and cultural practices. They argue that identities are closely related to structural features of the society and positional identities are connected to social affiliation and the way cultural systems work.

DEVELOPMENT OF THE RATIONALE OF THE STUDY

The project that this paper arises from set out to explore whether there is any difference between mechanical engineering (ME) and mathematics (M) degree students' conceptual development of the derivative concept over the first year of study and, if there are differences, to explore reasons for these differences. This study was conducted in a large university in Turkey. We do not claim that results from this study generalise beyond the confines of this university. The approach to data collection is naturalistic (Lincoln & Guba, 1985). Data were collected by a variety of means: quantitative (tests), qualitative (questionnaires and interviews) and ethnographic (classroom observations of semester 1 calculus courses and 'coffee-house' talk). In this paper we focus on two test items (see Figure 1) and provide summary data on pre-, post- and delayed post-tests and on classroom observations. The purpose of the summary data is to give the reader an appreciation of students' conceptions and their learning environments and also to inform the discussion section.

SUMMARY DATA

The pre-, post- and delayed-post tests were administered to 50 first year ME and 32 M students and addressed questions regarding, 'rate of change' and 'tangent' and were used to gain insight into: (a) how ME and M students' concept images of the derivative developed over the course, (b) how students dealt with rate of change and tangent concepts when questions were presented in graphic, algebraic and application forms and (c) whether there were any differences between ME and M students' performance in the different forms of these questions. The pre-test was administered to all students at the beginning of the course and there was no significant difference between ME and M students' performance. Both groups of students improved their performance in the post-test which was set at the end of the first semester. The delayed post-test was set towards the end of the second semester and both groups of students continued to improve. ME students, overall and in comparison with M students, did much better on rate of change-oriented test items regardless of whether the items were presented in algebraic, geometric or application-based forms. Similarly M students did much better than ME students on all forms of tangent-oriented questions. These trends (ME to rate of change, M to tangent) remained strong in the delayed post-test.

The calculus courses were observed and compared with students' notes to gain insights into which aspects of the derivative were 'privileged' in each department (Wertsch, 1991, p.124 uses 'privileging' in place of 'domination' to emphasise that a mediational means may be viewed as most appropriate in a particular setting). The analysis of observations and students' notes indicates (see, table 1) that ME students were taught more application (rate of change) aspects of the derivative compared to M students who were taught more theoretical or tangent-oriented aspects.

	Rate of change		**Tangent**	
	ME	M	ME	M
Duration examples	≈133 minutes (9 examples)	≈11 minutes (no examples)	≈10 minutes (no examples)	≈85 minutes (7 examples)

Table 1: The general analysis of ME and M calculus course' notes

DEVELOPMENT OF 'RATE OF CHANGE' VERSUS 'TANGENT' ITEMS

The trend (ME to rate of change, M to tangent) emerged from the post-test data. We decided to design two further items which might shed further light on the reasons behind this trend and administered these with the delayed post-test. Item 1 provided students with one rate of change-oriented question (A) and one tangent-oriented question (B) and asked them to state which one they would choose to solve if they were asked in their examination. When students chose each question, they were also asked to explain reasons behind their selection. Item 2 provided students with an imaginary situation where two students exchange ideas regarding their understanding of the derivative concept. Students were asked which view was closer to their way of thinking. These items were given to 45 ME and 32 M students.

Item 1: If the following two questions (A and B) were given in an examination and you only had to solve one of them, which one would it be? Please tick just one option and explain why you chose that one.

A.) At a certain time (t, seconds) the rate at which water flows (m^3/sec) into a water tank is given by the formula $f(t)= \frac{t^2}{4} +24t+125$. Find:

a.) The initial amount of water in the tank and its initial rate of change?
b.) What is the rate of change of flowing water at any time, t?
c.) The time at which the rate of change is $32\ m^3/sec^2$.

B.) Find the solutions of the following questions:

a.) Verify that the gradient of the tangent to the curve $y=x^2$ at a point $(x_1, x_1^2)=2\,x_1$.
b.) Find the equation of the tangent to the curve $y=2x^2-x+3$ which is parallel to the line $y=3x-2$.
c.) Show that the graph of $f(x)= x^{1/3}$ has a vertical tangent line at (0,0) and find an equation for it.

Item 2: Two university students from different departments are discussing the meaning of the derivative. They are trying to make sense of the concept in accordance with their departmental studies.
Ali says that "Derivative tells us how quickly and at what rate something is changing since it is related to moving object. For example, it can be drawn on to explain the relationship between the acceleration and velocity of a moving object.
Banu, however, says that "I think the derivative is a mathematical concept and it can be described as the slope of the tangent line of a graph of y against x".
a.) Which one is closer to the way of your own derivative definition? Please explain!
b.) If you had to support just one student, which one would you support and why?

Figure 1. Two items to explore reasons for rate of change and tangent orientations

RESULTS

We first present quantitative data (frequency counts) and then a categorisation of students' reasons for their choices. Tables 2 & 3 show students' responses to items 1 & 2.

Item 1	ME	M
Question A (A)	60 % (27)	22 % (7)
Question B (B)	40 % (18)	78 % (25)

Table 2. Students' responses (percentages–raw frequencies in brackets) to item 1

	ME		M	
Item 2	Item 2a	Item 2b	Item 2a	Item 2b
Ali (A)	51 % (23)	49 % (22)	19 % (6)	13 % (4)
Banu (B)	27 % (12)	49 % (22)	63 % (20)	78 % (25)
Both (A &B)	22 % (10)	2 % (1)	16 % (5)	3 % (1)
Not Attempted (NA)	0	0	3 % (1)	6 % (2)

Table 3. Students' responses (percentages–raw frequencies in brackets) to item 2

For item 1 ME students show a preference (60:40) for rate of change-oriented item over tangent-oriented one whilst M students show quite a strong preference (78:22) for tangent-oriented question. Similar preferences can be seen in item 2a. In item 2b the preference of the M students remains but the ME students are equally divided.

Tables 4 and 5 present a categorisation of students' reasons for their choices in items 1 & 2. Repeated reading of students' responses generated three emergent categories: Affiliation; Practice; and Ease. 'Affiliation' is a construct used by Nasir & Saxe (2003) to describe identification with a common cultural ancestry and distinctive cultural patterns and it appeared to us an appropriate term to apply to M and ME students who identified themselves as belonging to a particular department. 'Practice' here concerns students' calculus practices and this, as these students are novice practitioners, is related to what goes on in calculus courses. We use the term 'wider practices' in the discussion section when we attend to departmental features that are not solely concerned with calculus. 'Ease' here means what particular students reported that they found easy (not our decisions on the ease of items). We first

explain how we allocated students responses to these categories and present examples of students' responses for each category.

Affiliation (ME) ME students' responses were placed in this category when they mentioned any of the following: real life; applications; rate of change; engineering.

Student 1 *'What Ali says is closer. Calculating rates of change seems to me more real. On the other hand what Banu says is not far away, even it is so close. ... But since I am going to be an engineer, Ali's idea would be just different. Because I would be the one who makes mathematics concrete'.*

Affiliation (M) M students' responses were placed in this category when they mentioned any of the following: the exact nature of the derivative; the slope of a tangent; belonging to a mathematics department; interpretation from a mathematician standpoint; the comprehensiveness of the definition.

Student 2 *'Banu interprets the derivative from a mathematician's perspective, and Ali interprets it from a physicist standpoint. At the end of the day, since I too am from mathematics department, I find Banu's explanation closer to myself. But in essence they both present the essence of mathematics'.*

Practice (ME) ME students' responses were placed in this category when they mentioned the way calculus is being covered and used in their department.

Student 3 *'We are using it in that way and learning it that way'.*

Practice (M) M students' responses were placed in this category when they mentioned not knowing much about rate of change or the way calculus is being covered in their department.

Student 4 *'We are learning in that way and I don't know much about rate of change'.*

Ease(ME & M) Students' responses were placed in this category when they mentioned the 'ease' of this way of thinking about the derivative.

Student 5 *'Because it is easier'*

Categorisation of responses	**Engineers choosing A**			**Mathematicians choosing B**		
	Item1 60% (27)	Item 2a 51% (23)	Item 2b 49% (22)	Item1 78% (25)	Item 2a 63% (20)	Item 2b 78% (25)
Affiliation	20% (9)	44% (20)	44% (20)	13% (4)	47% (15)	66% (21)
Practice	11% (5)	7% (3)	4% (2)	50% (16)	28% (9)	13 % (4)
Ease	29% (13)	0	0	16% (5)	0	0

Table 4 Responses of ME students who chose As and M students who chose Bs

'Affiliation', 'practice' and 'ease' are cited by both groups of students for item 1 and there is no clear pattern to these responses but note that 50% of M students cite 'practice'. 'Ease' is not really applicable for item 2 and all students cite either 'application' or 'practice' with 'affiliation being the dominant stated reason.

Categorisation of responses	Engineers choosing B			Mathematicians choosing A		
	Item1 40% (18)	Item 2a 27% (12)	Item 2-b 49% (22)	Item1 22% (7)	Item 2a 19% (6)	Item 2-b 13% (4)
Affiliation	0	18% (8)	49% (22)	6% (2)	19% (6)	10 % (3)
Practice	9% (4)	9% (4)	0	0	0	0
Ease	31% (14)	0	0	13% (4)	0	0
Not categorised	0	0	0	3% (1)	0	3% (1)

Table 5 Responses of ME students who chose Bs and M students who chose As

'Ease' is the dominant cited reason in item 1 and 'affiliation' is the dominant cited reason in item 2 for students who do not follow the 'ME – rate of change, M – tangent' trend. That 49% of ME students' responses (item 2b) is with a mathematician's concept of the derivative is noteworthy.

DISCUSSION

In this section, we attend to possible reasons for ME students' tendency to rate of change and M students' tendency to tangent aspects of the derivative concept. We focus on practice and affiliation as these constructs, in the categorisation of students' responses, apply to both items 1 & 2 (see tables 4 & 5). We first discuss practice, then affiliation, then the relationship between affiliation and positional identity and end with questions for further research.

Table 1 shows that the ME students' calculus course 'privileged' rate of change examples whilst the M students' calculus course 'privileged' tangent examples. Calculus practices in each department are likely to have played an important role in the growth of the different tendencies in these two groups of students' performances In this regard, Kendal & Stacey (2000) show how students' conceptions of the derivative are strongly influenced by the aspect of the derivative privileged by their teachers. It is thus reasonable to infer that privileging of rate of change and of tangent aspects of the derivative in the two calculus courses influenced students' orientation and knowledge development.

Table 4 shows that both groups of students, but especially M students, referred to practice in explaining the reasons behind their choices in items 1 & 2, i.e. their preferences for rate of change or tangent forms of the derivative. Others, however, related their preferences for specific forms of the derivative to affiliation and ease (especially ME students). We do not believe that it is possible to separate students' feelings of affiliation and perceptions of ease from practice. We further believe that the specific calculus practices of the students, e.g. the type of examples they are most often presented with, are interrelated to the wider practices of their departments and that students' ways of participating are adapted to the constraints and affordances existing in each department (Greeno, 1998; Boaler & Greeno, 2000), i.e. these wider practices facilitate student access to some forms of the derivative but simultaneously constrain (maybe inadvertently) access to other forms.

Do the calculus practices alone explain ME students' proclivity to rate of change and M students' proclivity to tangent aspect of the derivative concept? Students responses suggest that the answer is 'no'. Affiliation (see tables 4 and 5) is the highest cited reason given by both groups of students in explaining their preferences for forms of knowledge. We interpret this as evidence for a developing personal association (being an engineer or a mathematician) towards particular conceptual forms of the derivative. The applicability of the derivative concept, for instance, was cited as a reason for their preferences for rate of change-oriented items by many ME students. In a similar manner many M students attached considerable importance to the 'exact' nature of the derivative. It is, therefore, plausible to infer that the reasons behind the students' different inclinations for the different forms of the derivative arise not only from the fact that their calculus practice privileged these forms but also because students from each department developed different affiliations towards different aspects of the derivative.

Affiliation appears to be an important construct in understanding undergraduate students' conceptions, but where does affiliation come from? We have noted that the ME to rate of change and M to tangent trend was not present in the pre-test but was present in the post-test. It appears to have developed during the first semester. We believe that Holland et al.'s (1998) notion of positional identity is relevant in explaining the genesis of affiliation. Students' developing affiliation towards particular forms of the knowledge could come into being as a result of the way that students comprehended and enacted their positions in the department to which they belong. This positioning can take many forms, mental and physical, e.g. going to the part of the mathematics section of the library that deals with applied (or pure) mathematics. In the course of their studies it is likely that students from each department began to position themselves according to their professional perspective and this positioning influenced their proclivity towards specific forms of knowledge. The concept of positional identity helps us to appreciate that what we have termed affiliation is not just a personal mental construct but, to use Holland et al.'s (1998) language, is shaped by enacting their positions. Many ME students, for example, attended meetings organised by the ME department with local professional engineers.

Positional identity is clearly a related construct to affiliation, but what else can be said about the relationship between these two? We cannot give definitive answers as our data is inconclusive but there are interesting questions for further research. We believe the relationship is a dialectical one, i.e. that the positions enacted by students help shape departmental affiliations and that students' affiliation is a characteristic of their emerging positional identities. Further to this, and with regard to students' emerging identities and their relationship with forms of knowledge, what relation exists between students' knowledge development and their emerging positional identities?

CONCLUSION

Mechanical engineering students develop a proclivity for rate of change aspects of the derivative whilst mathematics students develop a proclivity for tangent-oriented

aspects. Further to this, it appears that students' conceptual development of the derivative and the way they build relationships with its particular forms are closely related and co-evolve in accordance with departmental perspectives.

It has been argued that this difference between the conceptions of the both groups students cannot solely be attributed to the practice of the courses that the students followed. Departmental affiliation appears to have influence on cognition and play a crucial role in the emergence of this difference. The concept of positional identity has links with the concept of affiliation. An implication for further research into students' understanding of calculus at the undergraduate level is that researchers should not ignore students' departmental affiliation.

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ELEMENTARY GRADES STUDENTS' CAPACITY FOR FUNCTIONAL THINKING[i]

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This is a study of how urban elementary grades students develop and express functions. Data were analyzed according to the forms of representations students used, the progression in students' mathematical language and the operations they employed, and how they attended to one or more varying quantities. Findings indicate that students are capable of functional thinking at grades earlier than perhaps thought. In particular, data suggest that students can engage in co-variational thinking as early as kindergarten and are able to describe how quantities correspond as early as 1st-grade. Although pattern finding in single variable data sets is common in elementary curricula, we conclude that elementary grades mathematics should extend further to include functional thinking as well.

BACKGROUND FOR THE STUDY

Research increasingly documents the ability of elementary grades (PreK-5) students from diverse socioeconomic and educational backgrounds to engage in algebraic reasoning[ii] in ways that dispel developmental constraints previously imposed on them (e.g., Bastable & Schifter, 2003; Blanton & Kaput, 2003; Carpenter, Franke, & Levi, 2003; Carraher, Schliemann, & Brizuela, in press; Dougherty, 2003; Kaput & Blanton, in press; Schifter, 1999; Schliemann, Lara-Roth, & Goodrow, 2001). One of the forms algebraic reasoning takes involves *functional thinking*, which Smith (2003) describes as "representational thinking that focuses on the relationship between two (or more) varying quantities" and for which *functions* denote the "representational systems invented or appropriated by children to represent a generalization of a relationship among quantities". As reported earlier (Blanton & Kaput, 2002), our interest in the development of algebraic reasoning in elementary school mathematics led us to identify design aspects of tasks that might be used to exploit algebraic ideas, particularly in tasks where algebraic reasoning occurred through generalizing from numerical patterns to develop functional relationships. This study extends that work and builds on the emerging research base in early algebraic thinking by examining how students in elementary grades are able to develop and express functional relationships.

METHODOLOGY

The data for this study were taken from GEAAR, a 6-year, teacher professional development program in an urban school district designed to help teachers transform

their instructional resources and teaching practices to build on classroom opportunities for algebraic reasoning. We base the particular findings reported here on PreK-5 student responses from one of the district's schools to the task "Eyes and Tails". The task involves developing a functional relationship between an arbitrary amount of dogs and the corresponding total number of eyes or the total number of eyes and tails:

> **Eyes and Tails:**
>
> *Suppose you were at a dog shelter and you wanted to count all the dog eyes you saw. If there was one dog, how many eyes would there be? What if there were two dogs? Three dogs? 100 dogs? Do you see a relationship between the number of dogs and the total number of eyes? How would you describe this relationship? How do you know this works?*
>
> *Suppose you wanted to find out how many eyes and tails there were all together. How many eyes and tails are there for one dog? Two dogs? Three dogs? 100 dogs? How would you describe the relationship between the number of dogs and the total number of eyes and tails? How do you know this works?*

"Eyes and Tails" was selected because its accessibility across the grades allowed us to look for longitudinal trends in students' functional thinking. Student responses were collected from written work and teacher interviews and were analyzed by grade according to the types of representations students used at different grades, the progression of mathematical language in students' descriptions of functional relationships, how students tracked and organized data, the mathematical operations they employed to interpret functional relationships (i.e., additive vs. multiplicative), and how they expressed variation among quantities.

RESULTS

Pre-kindergarten (Ages 3–5)

The teacher and students spent time with paper cutouts of dogs, counting their eyes and tails. Students described the amounts as "even" or "odd". With the teacher's guidance, the whole class used a t-chart to organize their data. As a class, they recorded that one dog had 2 eyes and 1 tail, or a total of 3. They also determined that 2 dogs had 4 eyes and 2 tails, or 6 total. As the children offered the number of eyes, the teacher wrote that number in the appropriate box in the t-chart and put a corresponding number of dots below the box. She recorded the number of tails for a given number of dogs in a similar manner.

In finding the total number of eyes and tails for a given number of dogs, the teacher pointed to each of the dots as the class simultaneously counted. When the teacher asked about 3 dogs and 4 dogs, the children counted the number of eyes and tails using dog pictures on the floor in front of them. No predictions were made at this

grade and answers were determined by counting visible objects. Moreover, there was no indication from the data that students looked for patterns. However, we maintain that a significant mathematical event for these students was not only the development of correspondence between numeral and object, but also the introduction of a function table (t-chart) as a means to organize quantities that co-vary. The latter event reflects the early development of representational infrastructure to support algebraic reasoning.

Kindergarten

In one class, students recorded data by making a dot for each eye and a long mark for each tail. Dots were grouped in pairs or in 4-dot, 2x2 arrays. Dots (eyes) and marks (tails) were recorded under the number sentences that represented the total number of eyes and tails for a given amount of dogs. Data were aggregated by groups of dogs, which were drawn and painted by students, and the corresponding number sentences for total eyes or total eyes and tails, as well as dots and marks representing eyes and tails, were recorded and encircled by students (see Figure 1). Data were calculated for up to 10 dogs. T-charts were used in some kindergarten classes (with data recorded on the charts by the teacher) and some students identified the pattern in the amount of eyes as "counting by 2s", "more and more", and "every time we add one more dog, we get two eyes".

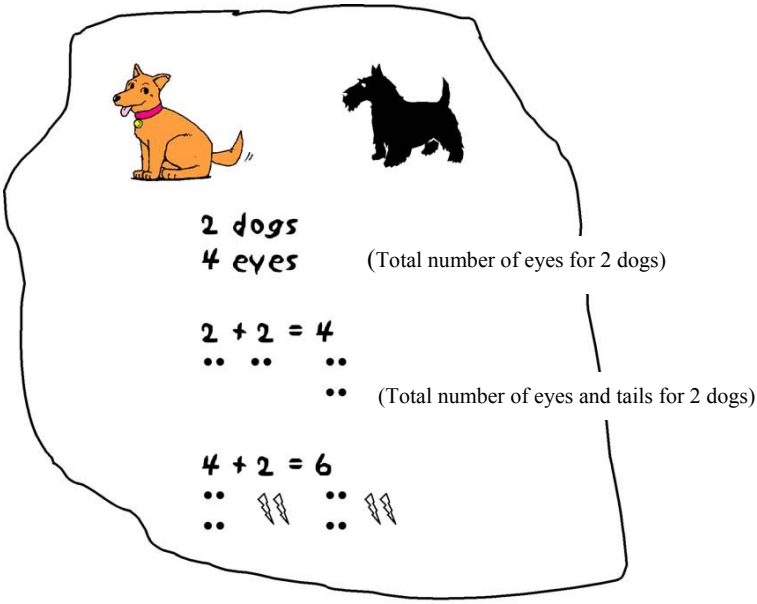

Figure 1. Kindergarten students' representation for 2 dogs.

In one class, after the teacher and students built a t-chart that recorded the number of eyes on 1, 2, 3 and 4 dogs (the teacher recorded the data), the following exchange occurred in which the teacher called on students to identify a pattern based on parity in the data:

Teacher: What if [we have] 5 dogs? Odd or even?

Student: Even.

Teacher: Why?

Student: We're skipping all the odd numbers.

We include this particular transcript here because it illustrates an important point. By asking students to analyze the data in terms of parity (even or odd) and not just quantity, the teacher required a further abstraction in student thinking. We find it mathematically significant that *kindergarten* students were not only able to recognize even and odd numbers (a concept we had observed as difficult for some *third-grade* students during the early stages of GEAAR), but were also able to articulate a pattern, albeit primitive, about parity in the data.

First grade

First-grade teachers noted that students had used t-charts previous to "Eyes and Tails". Moreover, students, rather than the teacher, recorded data on t-charts. They described patterns in the case of counting eyes and tails as "we are counting by 3s". Literacy activity was integrated into the problem in one class, where students made rhyming words and constructed poems in conjunction with the pattern "counting by 2s". With the teacher, students in this class tried to predict the number of eyes 7 dogs would have and used skip counting to find the answer. Students saw that the pattern would "double" (for total eyes), and then "triple" (for total eyes and tails).

Second grade

Students in one 2nd-grade class recorded their data on a t-chart for 1 to 10 dogs and were able to give a multiplicative relationship using natural language ("You have to double the number of dogs to get the number of eyes"). They then used this to predict the number of eyes for 100 dogs without counting the eyes. They constructed a similar t-chart for counting eyes and tails and used this, based on the information recorded in their t-charts, to predict that the total number of eyes and tails for 100 dogs would be 300.

Third, fourth, and fifth grades

Students in 3rd-grade classes used t-charts fluently, were able to express the rule multiplicatively in words and symbols, and could predict the number of eyes or eyes and tails for 100 dogs using their rule. In counting the number of eyes, students noted that "It doesn't matter how many dogs you have, you can just multiply it by 2". Students were able to describe this relationship as '$n\times 2$' and '$2\times n$'. One 3rd-grade class

graphed their results comparing the number of eyes with dogs (see Figure 2) and comparing the number of eyes and tails with the number of dogs. Fourth- and 5th-grade student work was similar to that in 3rd-grade, with the only noticeable difference being that students in later grades needed less data (only up to 3 dogs) to develop a function.

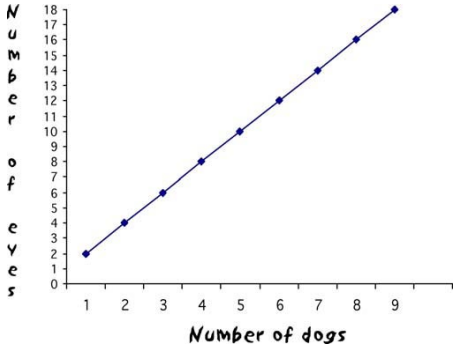

Figure 2. Third-grade students' graphical representation of the total number of eyes versus number of dogs.

DISCUSSION

These data indicate that very young learners are capable of functional thinking and suggest how that thinking might progress over grades Pre-K–5. Particularly, shifts occurred in how students were able to (1) use representational forms such as t-charts, (2) articulate and symbolize patterns, from natural language descriptions of additive relationships to symbolic representations of multiplicative relationships, and (3) account for co-varying quantities. The following discussion details that progression.

The development of representational infrastructure and students' symbol sense

Across the grades, students used tables, graphs, pictures, words and symbols to make sense of the task and to express mathematical relationships. Regarding the scaffolding of these representational forms, teachers were typically the recorders for t-charts in earliest grades, although by first grade students began to assume responsibility for this. In one kindergarten class, students did record the data on a class chart, but the teacher played a large role in organizing the data. By 2nd- and 3rd-grades, students seemed to use this representational tool fluently.

In grades Pre-K through 1, students relied on counting visible objects, keeping track of their counting in various ways through t-charts or making dots and marks for eyes and tails (see Figure 1). In early grades, t-charts became opportunities to re-represent marks with numerals as children worked on the correspondence between quantity and numeral representation. T-charts were the most common way, especially from 1st-grade through 5th-grade, that students organized and tracked data.

We observed that, by 3rd-grade, students were able to symbolize varying quantities with letters, and they seemed to have an emergent understanding of what these symbols represented. (We did note some confusion as to when a variable represented the number of dogs versus the number of eyes (or eyes and tails).) Moreover, third-grade students could express relationships in symbolized form (e.g., "number of eyes is *2n"*), although they did not fully symbolize the relationship in a form such as '*f(n)=2n*'. In 4th grade, some students wrote '□ × 3 = *n*' after constructing a t-chart. Although students primarily used words and symbols to describe the function, one 3rd-grade class did construct a line graph representing the number of dogs versus the total number of eyes (see Figure 2).

Finally, the ways students labelled their t-charts reflected increasingly sophisticated language. In grades PreK-1, t-chart headings were described in words ('dogs'; 'eyes'); in 2nd-grade, t-charts were labelled as "number of dogs" and "number of eyes". By 3rd-grade, symbols such as 'D' and 'E' were used for the number of dogs and eyes.

All of this suggests that teachers were able to scaffold students' thinking from a very early age so that diverse representational and linguistic tools became an increasing part of students' repertoire of doing mathematics.

How students accounted for varying quantities

Although finding patterns and predicting future values seemed understandably tentative in grades Pre-K–1, there were notable instances of this, such as the protocol recorded earlier in which kindergarten students found an "even" pattern in their data. When using a t-chart, 1st-grade students noticed patterns in how the number of eyes varied, and they described patterns in everyday language using both additive relationships ("we are counting by 3's) and multiplicative relationships ("double" and "triple"). Skip counting seemed to be the most common process for finding unknown values, and additive relationships were more common than multiplicative ones. By 2nd-grade, students were able to articulate a multiplicative relationship using everyday language ("You have to double the number of dogs to get the number of eyes") and use this to predict the number of eyes for 100 dogs without counting the eyes. In later grades, students needed increasingly fewer data values to determine a functional relationship and make predictions.

What we found particularly compelling in the data was how early students began to think about how quantities co-varied. One kindergarten class described an additive relationship between the number of eyes and dogs as "every time we add one more dog we get two more eyes", indicating that they were attending to both the number of dogs and eyes simultaneously and were able to describe how these quantities co-varied. In 1st-grade, students identified a multiplicative relationship of "doubles" and "triples" to describe the number of eyes and the number of eyes and tails, respectively, for an arbitrary number of dogs. In 2nd-grade, students also saw a multiplicative relationship ("doubles"; "If you double the number of dogs you get the number of eyes"). The observation that the pattern "doubles" or "triples" indicates

that students were attending to how quantities corresponded. That is, some quantity needed to be doubled to get the total amount of eyes or eyes and tails. Since data in the 'output' column (e.g., total number of eyes; 2, 4, 6, 8…) were *not* doubled (i.e., $4x2 \neq 6$; $6x2 \neq 8$), this suggests that students were not looking 'down' the column of eye data (which would have resulted in a pattern of "add 2 every time" or "count by 2's"), but 'across'. By 3rd-grade and beyond, students seemed fairly sophisticated in their ability to attend to how two quantities varied simultaneously and to symbolize this relationship as a functional correspondence (e.g., "the number of eyes = $2n$").

Although elementary grades mathematics has in more recent years included notions of patterning, it has not traditionally attended to functional thinking, especially in grades Pre-K–2. Yet, from our analysis, we found that students could engage in co-variational thinking as early as kindergarten and were subsequently able to describe how quantities corresponded as early as 1st-grade. More abstract symbolizing using letters as variables occurred as early as 3rd-grade. We conjecture that the typical emphasis on pattern finding in single variable data sets in early elementary grades might impede an emphasis on functional thinking in later elementary grades and beyond. In particular, there was evidence that when 1st-grade students engaged in functional thinking, they were sometimes redirected to an analysis of a single variable (e.g., finding a pattern in the total number of eyes). This focus could be a habit of mind engendered in teachers by existing curricula. Ultimately, pattern finding in single variable data has less predictive capacity and is less powerful mathematically than functional thinking. There is a fundamental conceptual shift that must occur in teachers' thinking in order to move from analyses of single variable data to those attending to two or more quantities simultaneously. As a result, we suggest that curricula for grades PreK-5 should attend to how two or more quantities vary simultaneously, not just simple patterning. This study supports the claim that young students have the capacity for this type of functional thinking.

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[ii] While algebraic reasoning can take on various mathematical forms in the classroom (Kaput, 1998), we see it broadly as a habit of mind that permeates all of mathematics and that involves students' capacity to build, justify, and express conjectures about mathematical structure and relationship.

FOR THE SAKE OF THE CHILDREN: MAINTAINING THE MOMENTUM OF PROFESSIONAL DEVELOPMENT

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This paper reports the survey findings from a study designed to evaluate the impact of a large-scale professional development program for primary mathematics teachers. While a number of aspects of the program were considered potential agents for promoting on-going learning in teachers, others emerged as significant barriers to its maintenance. What seems to emerge as a driving force of the program was the perception by teachers that it would ultimately benefit the children.

There is now an extensive body of research-based literature concerned with what makes professional development of teachers effective. This literature provides guidance for the establishment of good quality professional development (Loucks-Horsley, Hewson, Love, & Styles, 1998). Key concerns for teacher professional development programs have also been identified. These include the desire for sustained teacher change and on-going learning. Despite the extent of the literature and the identification of such key concerns, there is relatively little systematic research concerned with what ongoing teacher learning looks like and how it can be achieved (Garet, Porter, Desmimone, Birman & Yoon, 2001). Some of these concerns have started to be addressed by a growing body of literature surrounding the Australian numeracy project, Count Me In Too (e.g. Bobis & Gould, 2000, Mitchelmore & White, 2003, Wright & Gould, 2002). The identification of factors responsible for maintaining the momentum of this large-scale professional development program for primary mathematics teachers was an overarching concern of the study reported here.

BACKGROUND AND KEY FEATURES OF COUNT ME IN TOO

> What is happening in mathematics teaching in New South Wales' public schools is truly exciting. New South Wales is an international leader in its widespread translation of mathematics education research into practice.
>
> (Cobb, 2001)

Count Me In Too (CMIT) is a research-based professional development initiative of the government school system in the state of New South Wales (NSW), Australia. This large school system provides for a population of about seven million people, including approximately 1,700 primary schools. CMIT was piloted in 1996 in just 12 schools across the state under the name 'Count Me In', and has progressively grown in reputation and implementation with over 1600 primary schools having implemented the program by 2003. It has also been extremely influential on

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numeracy programs of other states in Australia and has been adopted nationally in New Zealand (Thomas & Ward, 2001).

From its conception, CMIT has involved the collaboration of government school system leaders and university-based researchers in mathematics education. The two main aims of the program are to professionally develop teachers so that they better understand young children's mathematical development (Stewart, Wright & Gould, 1998), and the enhancement of the mathematical achievement of young children. The emphasis of CMIT is on the advancement of children's mathematical solution strategies.

The CMIT model of professional development emphases long-term classroom-based learning and aims to establish a community of learners among four linked groups—"academic facilitators, consultants, teachers and students" (NSW Department of Education and Training, 2003, p. 2). To achieve this, 40 mathematics consultants working in designated school districts across the state, work with a group of teachers from a small number of schools over an extended period of time—normally 10 to 20 weeks. During this period, consultants support teachers to acquire skills in diagnostic interviewing, and develop their understanding of a research-based Learning Framework in Number (henceforth referred to as the *Framework*) (Wright, 1998). The *Framework* is used by teachers to not only identify the level of development each child has attained but provides instructional guidance as to what each student needs to work towards. Further details about the *Framework* and the diagnostic interview can be obtained from Wright (1998).

AIM OF THE CURRENT STUDY

Initially, number knowledge in the first three years of school (Kindergarten to Year 2) was the main focus of CMIT and systematic research-based evaluations have indicated that the program has been successful (e.g. Bobis, 2001; Mitchelmore & White, 2003). However, as the program moved into the subsequent years of schooling (Years 3 and 4), the nature of support provided by consultants changed and government administrators showed concern for sustaining the changes to classroom practice that had occurred and for maintaining the momentum of the program's implementation. The major aim of the study reported here was to evaluate the program's implementation in Year 3 and 4 classrooms.

METHOD

The study gathered data from two different sources, namely the mathematics consultants and Year 3 and 4 teachers who had been involved in the CMIT program. Information was collected via a teacher survey, interviews and informal discussions with teachers and mathematics consultants. Teacher interviews and informal discussions were conducted as a result of three schools being selected for case study. Only data from the teacher surveys will be referred to in this paper.

Materials and procedures

The prime purpose of the teacher survey was to gain information about the perceived strengths and weaknesses of CMIT from a range of Year 3 and 4 teachers. It was a 3-page document comprising two main parts. Part A contained 8 questions designed to gain information about each respondent's school context and individual teaching background. Part B contained 15 questions designed to elicit individual teacher's reactions to various aspects of the CMIT program. Each question in Part B required an open-ended response. For example, Question 16 requested information about the barriers or challenges teachers perceived they would face when implementing CMIT in their classrooms in the future.

Surveys were distributed to teachers eligible to participate in the evaluation by their respective district mathematics consultant. Teachers were eligible to receive the survey if they (a) had completed the initial diagnostic testing of their students, and (b) had implemented CMIT lessons for at least five weeks. One hundred surveys were distributed.

Data from each survey were transferred to a text file. Each text file was then transported into a qualitative data analysis computer program, QSR NUD*IST (1997), to assist with analysis. Contextual and biographical data from Part A of the survey were collated using text searches. Open-ended responses to items in Part B of the survey were categorised into major themes and then coded for analysis.

RESULTS AND DISCUSSION

Despite 100 surveys being distributed, 108 were returned with representations from 20 of the 40 school districts across NSW. The extra surveys were the result of teachers copying and distributing it to colleagues. Due to either missing data or the lateness with which a number of the surveys were returned, only 95 were included in the final analysis.

Contextual and biographical data from Part A of the survey will be reported briefly so as to provide an indication of the nature of the sample. Open-ended responses to items in Part B of the survey will be reported using the major themes identified for each item. Given limitations of length, the discussion will focus mainly on what respondents perceived to be the most effective aspects of the program and the barriers to its successful implementation.

Contextual and biographical information

The sample of Year 3 and 4 teachers who responded to the survey was fairly representative of the general primary school teacher population in NSW, namely, the majority were female (83%) in the 41 to 50 age range (63%) with more than 21 years teaching experience (59%). Seventy-two percent of teachers who responded to the survey had been teaching Year 3 and/or 4 for more than 4 years and 37% of these had been teaching the same grade for more than 7 years. This indicates that the teachers

who completed the survey were an extremely experienced group, and particularly experienced at teaching Year 3 and 4 students.

The majority of teachers (71.6%) who responded to the survey had been implementing CMIT in their classrooms for only one year or less. Only 11.6% of teachers had implemented the program for two or three years. Hence, while experienced teachers, their experience with the CMIT program was still very limited.

Open-ended responses

Generally, teachers' responses to CMIT were very positive. Only 2 respondents indicated that, if the decision to implement CMIT were their own, they would select not to continue with it.

While commenting on the impact of the program, 69.5% of teachers considered their attitude to mathematics and the teaching of mathematics had improved as a result of their involvement in CMIT. Many teachers attributed the change to seeing the "children improve their skills" and "understanding the reason behind what we do". Others considered their attitudes had changed towards the "use of textbooks", "written algorithms" versus mental computation and "allowing games in the classroom". Nearly every teacher who considered their attitude toward mathematics had not changed as a result of their involvement in CMIT (14.7%) thought that the program merely confirmed their prior beliefs about mathematics and supported methods of teaching that they had always used.

Content knowledge in a variety of areas was considered to have increased by 48.3% of respondents. Some teachers considered that the "deeper understanding of the philosophy" surrounding CMIT gave them greater "ownership" and "understanding" of a broad range of content leading "to a greater interest" in mathematics. However, the majority of teachers highlighted an increased knowledge in specific aspects of mathematic content. For example, teachers mentioned their new knowledge about the importance of "arrays to teach multiplication and division", the "better understanding of place value" and how it "is integral to all number understanding". The most frequently mentioned area of content knowledge to improve related to mental computation. Many teachers considered that "it has affected the way I mentally compute now and I pass this on to the children" or that they were now "aware of the value of mental computation skills" and so emphasised this more in their classrooms. Other teachers considered that their knowledge "of what to teach had not changed, just how to teach it".

The majority of teachers considered that their understanding of how children learn mathematics (71.6%) and the way they taught mathematics (77.9%) had changed the most as a result of their involvement in CMIT. One teacher commented that "it's scary what I didn't know" about how children learn mathematics. The majority of responses made reference to a "better understanding of the developmental stages in children's thinking" and knowing "how to move them onto the next stage". This "better understanding" or "insight as to how children learn" and the "different

strategies children use" was usually a result of the diagnostic interview or their understanding of the *Framework*.

Reported changes to the way teachers taught mathematics varied enormously. However, there were 4 aspects that were mentioned more frequently. Foremost among them was the use of "more hands-on, fun games" that were selected on the basis of children's "strategy development". A second aspect mentioned regularly concerned the emphasis on thinking strategies. In particular, the use of multiple methods for mental computation was highlighted by teachers. For example:

> Now I ask them instead of telling them. I give them time to think and we have a very enjoyable and productive environment. (There is) much more sequential development of teaching number with a greater commitment to using a variety of strategies to encourage thinking mathematically.

The majority of teachers indicated that they considered CMIT to be "worthwhile" and therefore willing to continue with the program, but only 25.3% of teachers indicated that they were entirely satisfied with either their initial training or the follow-up support they received to implement the program. Fifteen percent of teachers considered their training ineffective with the remaining respondents indicating that they were only partially satisfied with the effectiveness of their initial training and support to implement CMIT. Teachers who indicated most satisfaction with their training and the manner in which they were implementing the program were those who had received considerable in-school support in the form of classroom visits from their district mathematics consultant.

When asked to comment on the most helpful aspects of their training, teachers identified 5 crucial features—the practical resources and activities, the assessment process, classroom support, the influence of significant people and the opportunity to share ideas. Practical resources and activities were highlighted by 38.9% of respondents as being extremely helpful during the program "as I could go straight back to my classroom and do them (even though I was still struggling with the conceptual framework)". The second most frequently cited aspect of the training considered helpful by teachers, was the assessment (diagnostic interview). While a number of respondents thought the main aim of the program was to introduce the assessment interview, 29.5% indicated that "learning how to assess" was the "most useful" aspect of their training. For example:

> …to find out how a child thinks and where they are up to. Then to teach to that, and assess again later. This helped me to become very familiar with the learning framework and the range (of abilities) in my classroom.

Another frequently cited aspect of the program considered to be most helpful to teachers was classroom support (22.1%). Classroom-based support in the form of demonstration lessons and class visits by consultants was only provided to a small number of Year 3 and 4 teachers during the introduction of the program. Despite this limitation, its effectiveness was acknowledged by the majority of those who received

it. Teachers considered the classroom support "brought the nuts and bolts (of the program) to life".

Aspects of the CMIT training considered ineffective or "least helpful" by teachers included "the overload of information" (13.7%). Teachers referred to the initial training days as "daunting", "crash courses" where they were "bombarded with paper and activities". Many teachers considered their training to be "too much to cope with at once" with "too little follow-up support". While some teachers indicated that they "struggled at first", they "eventually worked it out" but it was "stressful and time consuming" when they "did not know what it looked like to implement". Another aspect of the initial training that received heavy criticism from 27.4% of teachers was the lack of a "systematically organised folder of activities and resources". Teachers "felt overwhelmed" by the need and "time required to make so many resources". While the practical ideas and resources introduced during initial training days were perceived to be a positive aspect of the training by 38.9% of teachers, the initial production and implementation of them was perceived negatively by an equal number of respondents.

When considering the challenges or barriers to the implementation of CMIT in their classrooms, 45.3% of teachers referred to issues of "time". This is consistent with previous evaluations of CMIT (Bobis, 1996; 2000), where the problem of *not enough time* is regularly raised by teachers. In the current study, teachers considered there to be a lack of time "to meet" with other teachers "to gain new ideas", to "complete the testing", "to make the resources", "to think of different ways to utilise the same resources", "to do the grouping", "to teach the activities", "to maintain and organise the resources", or "time to feel comfortable with the program and feel a sense of direction". While *lack of time* was the most commonly cited challenge to the implementation of CMIT, a number of teachers acknowledged that their concerns would be reduced in subsequent years of its implementation once the initial resources were made and they had become more familiar with the assessment procedures.

The second most frequently cited challenge facing teachers related to resources (31.6%). While some of the problems concerned the "time" for making and maintaining them, other issues included: "becoming familiar with all the materials available", "having easy access to resources", "having enough funds to purchase the necessary resources", "having enough resources for each teacher to avoid sharing", "having enough activities for all the children" so they "don't get bored with the same ones", and "getting the resources organised".

Class management issues were mentioned by 25.3% of teachers as presenting a challenge to their implementation of CMIT. They included problems associated with "not knowing what it looked like in the classroom", "ensuring that all children are learning from the group activities and not letting others do the thinking for them", the increased "noise" level due to group work and "management of multiple levels/games in the class". In addition, a number of teachers considered large class

sizes and lack of space to do group work a barrier to their implementation of the program.

SUMMARY AND CONCLUSION

The reporting of results in this paper have focussed on what respondents perceived to be the most effective aspects of a large-scale professional development program for primary mathematics teachers and the barriers to its successful implementation. While aspects of CMIT considered most effective included the practical resources and activities, the assessment process, the influence of significant people, classroom support and the opportunity to share ideas, a number of factors emerged as significant barriers to teachers' implementation of the program. These factors predominantly related to issues of time, resources, class management and information overload.

Inherent in the reporting of these findings is the concern for the identification of factors likely to maintain the momentum of the CMIT program. What factors are more likely to sustain teacher change and promote on-going learning? A number of CMIT features are potential agents for such growth in teachers. For instance, one respondent commented:

> What a change—a program which supports students and teachers at the same time—that's how we create life-long learning.

A number of teachers volunteered concluding comments regarding their overall opinion of CMIT at the end of the survey. Despite a high proportion of teachers indicating some significant issues with the implementation of the program, 30% of teachers communicated their intentions to continue with its implementation mainly because they considered it would ultimately benefit their students. This sentiment is characterised by one teacher's comments:

> It is taking time for some teachers to change habits and attitudes of 20 years—but they are willing to have-a-go as long as there is support and they can see it benefits their students.

Hence, the factor that seems to emerge as assuming greater significance than concerns surrounding issues of time, resources and the like, is teachers' inherent perception of the program's worth for children.

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FROM FORMAL TO SEMI-INFORMAL ALGORITHMS: THE PASSAGE OF A CLASSROOM INTO A NEW MATHEMATICAL REALITY

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In contrast to the traditional top- down approach, a bottom-up approach is proposed by current reform in mathematics education. According to this alternative proposal, algorithmizing is the activity in which students should be involved. What can we do when we want to enact such an algorithmizing approach in a classroom and our students have already been instructed the algorithms in a traditional way? The students have to move from a school-based to an inquiry-based mathematical reality. Is the passage from one reality to the other so easy? The focus of this paper is on the difficulties that a fifth-grade classroom met as we tried to revisit the multiplication and division algorithms, which had been taught in a traditional way. How these difficulties influenced the emergence of mathematical content?

INTRODUCTION

When we conduct a classroom teaching experiment, our general purpose is to attempt a change of the "school mathematics tradition". An "inquiry mathematics tradition" is what we are looking for (Cobb et al., 1992). Such a change implies that with our support students will come to experience mathematics in a different way. From a mathematical reality where students comprehend mathematics as a set of ready-made propositions and procedures, a new inquiry-based mathematical reality where mathematics is viewed as a human activity has to emerge.

According to Mehan & Wood (1975), realities are permeable and so the passage to a new reality is feasible. However, this passage is fraught with difficulties if we take into account their characterization of realities. As they claim, realities are sustained through the reflexive use of bodies of knowledge in interaction. Reflexivity means that a reality is not easily abandoned. Even when counter evidence is provided, it reflexively becomes evidence for the sustenance of an assumed reality. It is then interesting to investigate the difficulties that may occur as the members of a classroom are guided towards an inquiry-based mathematical reality.

Students in a fifth-grade classroom had already memorized the steps of the formal multiplication and division algorithms. Now they have to revisit these algorithms in order to extend their practice to numerals with any number of digits. In previous grades, instruction had been based on the use of concrete representations accompanying the explanation of numerical examples. Drill and practice always preceded the application of the algorithms to the solution of problems. Lack of proficiency and insight, as well as low applicability are usually mentioned as the

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negative effects of this approach (Hart, 1981; Resnick & Omanson, 1987). Students come to identify understanding to following the teacher's or the textbook's procedural instructions to obtain correct answers. On the other hand, as algorithms are taught out of context, students do not know when it is appropriate to apply them. A school mathematics based reality was thus well established by the students of this classroom.

To avoid the negative effects of this top-down approach, an alternative bottom-up approach would be to support students' construction of algorithms based on their own activity. Starting from contextual problems students can generate their own procedures. Through shortening and schematizing, these procedures can take the form of the conventional algorithms. Even if students don't reach formal algorithms, the quality of their understanding would counterbalance the development of semi-informal algorithms (Gravemeijer, 2003). In this way, algorithmizing becomes the main practice students are involved in (Freudenthal, 1991). The reinvention of algorithms makes students' mathematical reality inquiry-based.

For the students of the above-mentioned classroom the bottom-up teaching approach had to be adjusted, if we wanted them to experience an inquiry-based mathematical reality. As we had to revisit algorithms with these students, their difficulties in establishing such a reality would be more easily investigated. These difficulties would be greater due to their instrumental understanding of algorithms. The focus of this paper will be to examine the role of these difficulties as they influence the emergence of mathematical content in this classroom. In this process, the relations between the old and the new reality will come to the fore and the passage into a new mathematical reality will be illuminated. Apart from its theoretical importance, our attempt has also practical implications. Teachers intending to develop a bottom-up teaching approach for algorithms cannot usually enact it. Students' prior experiences inhibit their attempts and so they are skeptical about their teaching effectiveness.

THEORETICAL FRAMEWORK

Studying the difficulty of the passage of a traditional classroom into an inquiry-based mathematical reality, the emergent perspective on classroom life is of relevance (Cobb & Yackel, 1996). Any mathematical reality "is becoming" through the coordinated efforts of the individual students as they participate in diverse ways in the communal mathematical meaning-making activities of their classroom. Students' taken-as-shared meanings emerging through their own interactions can be considered as the building blocks for the construction of their reality. In turn, this reality may constrain or enable their individual constructions. The reflexivity between the individual and social aspects of their mathematical activity in constructing meanings for the multiplication and division algorithms will be assumed as we analyze individual students' contributions.

METHODOLOGY

Our data is based on a fifth grade classroom in a public school of Athens at the beginning of the school year 2003-04. It consists of 15 video recorded lessons, the 22 students' worksheets and written tests. The presenting author taught most of the lessons, aiming in students' understanding of multiplication and division algorithms. Development of students' multiplicative reasoning about quantities was considered necessary both for understanding the algorithms as well as the subsequent unit that focused on fractions.

For this purpose, we prepared an initial set of instructional activities based on Gravemeijer's (1998) heuristic of emergent models. A learning path has been anticipated through which students could be supported in developing insightful ways of reasoning with algorithms. More specifically, we expected that students' reasoning with a ratio table would emerge as a model of informal solutions to multiplication and division problems. Eventually, we anticipated that reasoning with the ratio table (see results section below) would serve as a model for the construction of semi-informal algorithms and would provide opportunities for our students' developing interpretations of the formal algorithms. However, it must be noted that the set of activities used in our classroom would be tailored to the students' needs. Their actual trajectory may not match our hypothetical learning path.

As we conduct our analysis, the construct of classroom mathematical practice developed by Cobb and his colleagues (Cobb et al., 2001) will be useful. This research group differentiates between three aspects of a mathematical practice: (a) a taken-as-shared purpose, (b) taken-as-shared ways of reasoning with tools and symbols, and (c) taken-as-shared forms of mathematical argumentation. We will be focusing on instances where individual students' ways of acting can be traced back to their school mathematics based reality. Their relationship to the above three aspects of a practice will allow us to understand the difficulties the passage to an inquiry-based mathematical reality entails. The delineation of mathematical practices as a means to describe the inquiry-based reality established in our classroom is not of our concern in this paper. However, one may notice that the instances we will be referring to may belong to different practices.

RESULTS

From formal algorithms to informal ways of operating

In our first lesson, students were asked to solve multiplication and division problems. What we noticed was that students: (1) did not have any different solutions to offer when solving a problem apart from using the standard algorithms, (2) were uncertain about which operation to perform, and (3) did not have any meaning for the steps of the algorithms used. No doubt that the school-based mathematical reality was overarching and constraining their activity. On the other hand, students appeared to have mastery of the multiplication and division formal algorithms.

For the next lesson, we distributed a worksheet with the solutions of hypothetical younger students on a multiplication problem. The problem on which students were invited to explain and justify the solutions was: “A bookcase has eight shelves. Each shelve has 23 books. How many are all of its books?” The intent of this task was to give students the opportunity to reflect on multiplicative relationships. The role of these relationships in building the algorithms might then be approached through properly designed solutions.

As an example, repeated doubling was used as a means to calculate the answer:

$$\begin{array}{rrr} 23 & 46 & 92 \\ \underline{+23} & \underline{+46} & \underline{+92} \\ 46 & 92 & 184 \end{array}$$

Below is the dialogue between the teacher and a student who was willing to explain the above solution:

1	T:	What did this child do? I mean how did she think?
2	S1:	Additions.
3	T:	Can you explain what did she do?
4	S1:	She added 23 and 23 and she found 46. Then she added once more…46 and 46 and she found 92. And then she again added 92 and 92 and she found 184.
5	T:	She found the same answer! But do you understand her way? Can someone else explain to us…why did she do these additions? [Students do not respond]

Initially, students were not in a position to see any connection between the above additions and the situation. Students merely read the additions. Searching for a reason behind a calculation was not a goal in their mathematical reality. Criteria for judging when an explanation would be appropriate were lacking. That is why their explanations were exclusively calculational.

Similar tasks along with our support (i.e. drawings, symbolizing their explanations on the board, etc.) helped students to start interpreting solutions in a multiplicative way. These interpretations were the best we could achieve from students deeply immersed in the school mathematics reality. The form of their arguments was getting a contextual character. Operating informally in multiplication and division situations was finally instigated.

Comparing the algorithms with carefully chosen exemplary solutions had also become a topic of discussion. Detection of their similarities and differences came as a result of these discussions. However, we should not forget that our students did not invent the solutions. These had been given ready-made. This significant deviation

from the bottom-up approach did not guarantee the re-construction of meaningful algorithms.

From informal to semi-informal ways of operating

To support our students' development of their own semi-informal ways of multiplicative operating, we introduced the model of ratio table in the classroom (National Center for Research in Mathematical Sciences Education & Freudenthal Institute, 1998). The opportunities our students had in interpreting multiplicatively solutions of hypothetical younger students might now be utilized. Reasoning with the ratio table could be based on this prior experience. Acting with this model insightfully was expected to ensure a meaning for the operations of multiplication and division, as well as for their algorithms. With these conjectures in mind, we told students about a fourth-grader who was used to organize his solutions with the help of a table. In the problem: "A crate of lemonades contains 24 bottles. If a supermarket buys 49 full crates, how many bottles has it bought?" this student's solution was presented as follows:

Crates	1	10	5	4	9	40	49
Bottles	24	240	120	96	216	960	1,176

Our students did not seem to have any difficulty understanding this fourth grader's reasoning with the ratio table. In the same lesson, starting from a multiplication problem, students recorded their different solutions on ratio tables and compared them, in terms of their efficiency. It was not but until a few lessons later, that a measuring situation involving the division 135:12, led a student to the following table:

Minibuses	1	3	5	135
Students	12	36	60	

X 3

X 5

Perhaps, the use of the ratio table was confounded with the use of place value tables like:

Hundreds	Tens	Units	The number
1	3	5	135

Other students did not strongly object his way of using the ratio table. In addition, it was not much later that similar solutions were given by a group of students in a written test. We were alerted by this instrumental use of the ratio table. It seemed that the use of this tool would come to have in this classroom a meaning related to the school based mathematical reality. Rather than students reasoning about quantities in multiplicative ways, they were looking for patterns in the numerals. From our perspective, the inquiry-based mathematical reality was at risk.

Asking students to anticipate the steps they would have to take, as they were using a ratio table, might help them to change the purpose of their activity. Questions like: "Which number are you looking for?" "Where is the unknown number going to be on your table?" and "Can you say in advance, the steps you intend to take?" were instrumental in reorienting students' actions with the ratio table. Through these questions students' activity was gradually focused on explaining the reasons for the steps they proposed.

From semi-informal ways of operating to semi-informal algorithms

Eventually, students could reason with the ratio table and solve a variety of multiplication and division problems. By the end of the instructional sequence, we surprisingly saw that there were students who were still choosing operations at random. For example, in a multiplication problem, they would try to divide the given numbers by using the division algorithm. For these students, the use of the ratio table did not evolve into a model for reasoning with the algorithms. Even if they could use this model, they could not relate it to the multiplicative relationships implied in the situation at hand. Apart from the inherent difficulty that such an undertaking involves, the vestiges of their old reality were still prevailing.

Encouraging students to estimate their answers did not prove to help students increase their awareness of structuring problems multiplicatively. To avoid the random selection of operations, we were inviting students not to be thinking of which operation to select. Starting their work from the ratio table did not encase them within the vicious circle of their school based mathematical reality. Reasoning with the ratio table came to constitute a semi-informal algorithm. Efficient and sophisticated solutions were commonly produced in our classroom. The problem: " For 5 days someone was paid 140 €. How many euros was he paid for each day?" could be solved by methods like:

Euros	140	280	28
Days	5	10	1

Euros of each day	1	2	10	20	8	28
Total amount	5	10	50	100	40	140

CONCLUSIONS

We tried to change the approach by which the multiplication and division algorithms had been taught in a fifth-grade classroom. A pure algorithmizing bottom-up approach was not feasible. Students already knew the algorithms and even more so they had constructed a school-based mathematical reality. The passage to an inquiry-based mathematical reality cannot be automatic. Students' old habits were coming to the fore and influenced the learning path of the classroom.

One may view these habits as inhibiting the enactment of an algorithmizing approach. In our classroom these difficulties functioned as opportunities to redesign the hypothetical learning trajectory we had in mind when we started our teaching experiment. However, students' development of semi-informal algorithms was our only alternative if we wanted students to walk away from their old reality. We should note that the quality of understanding in this classroom was not only a matter of the mathematical practices we tried to develop. The classroom social norms were also of our concern.

The influence of our students' school-based mathematical reality on their development of multiplicative reasoning declined. Their passage to an inquiry-based mathematical reality is still incomplete. At least, we hope that our students have already experienced the distinction between the two realities.

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A MEASURE OF RULERS - THE IMPORTANCE OF UNITS IN A MEASURE

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Recent publications continue to show that significant numbers of students in junior grades, while competent in basic ruler skills, seem not to have acquired important concepts about how rulers work and units of length. This paper reports on the results of a set of tasks completed by students from Grades 5 and 6. The results show that many students at the end of their primary schooling are unable to identify the unit of measure for length on a ruler or on the commonly used one centimeter cube. It is suggested that early measurement activities include explicit instruction in the relationships between informal units and the construction of scales on rulers.

Introduction

In recent years increasing attention has been paid to the teaching of measurement in mathematics lessons. Analyses of data gathered through regular large-scale testing by The National Assessment of Educational Progress (NAEP) show that while students have shown steady overall improvement in basic measurement skills and concepts since 1990 there appear to be significant gaps in student understanding of how scales on formal measuring tools work (Strutchens, Martin and Kenney, 2003). This becomes apparent when students are asked to measure lengths not aligned to zero or when the scale to be used has no numbers on it. Students seem not to have constructed adequate understandings of the property of length (Wilson and Rowland, 1993) and of the linear nature of units of measure (Bragg and Outhred, 2001). It is also apparent that while most students by Grade 5 appear competent with basic paper and pencil measurement and construction tasks many students are also unable to indicate *what is being counted* in the measurement process (Bragg and Outhred, 2000a).

Hiebert (1984) suggested that the discrepancy between procedural knowledge and conceptual knowledge may lie in the student's failure to link classroom experiences with the formal symbols. This may occur, for example, at the point where understandings about units of measure become represented in the markings and numerals on a scale (Stephan and Clements, 2003). "The hash marks and numerals on a ruler therefore, represent the result of iterating 12 inch-sized units" (p.4). Many students, however, may come to understand measuring length solely as an exercise in applying some rules for the alignment of an object and the reading of a number (Bragg and Outhred, 2001).

Bragg and Outhred (2001) showed that significant numbers of students in Grades 3-5 are unclear about what is being counted when they use one-centimetre cubes to measure length even though they were able to align and count them correctly. This is important because students in Grade 3 use the same cubes to measure area, perimeter

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and then volume. This confusion is also apparent when students are asked to indicate which feature of the scale on a ruler is counted when measuring a length. Younger students were more likely to colour in the spaces between the unit markers while older students were more likely to count the unit markers (hash marks) themselves. For these students "…the marks on the ruler "mask" the intended conceptual understanding involved in measurement" (Stephan and Clements, 2003, p.5). Some students simply did not believe that anything was counted at all, believing instead that the number at the end of the object was the measure.

The process of *iteration* is a fundamental concept that must be learned early in the measurement curriculum (Barrett, Jones, Thornton, and Dickson, 2003). Once a unit has been selected the measure obtained by counting tells how many of these units, placed end to end, are used to cover the length of the object. The tendency to count unit markers when the scale is unfamiliar would seem to indicate that students may have connected the iteration of informal units with the most prominent feature of on the ruler, the unit marker, even though the unit markers are at right angles to the *length* of the object or line being measured. This indicates that some students may understand the measurement of length using informal units and applying a ruler to be two separate skills. The first skill involves the correct use of a 'count-the-object or -action' process to determine a length while the second skill uses rules about the correct alignment and reading to obtain a measure with a ruler. It has therefore been suggested that teachers should not rely on paper-and-pencil tests of measuring as an indication that students have acquired a deep understanding of units and scales (Bragg and Outhred, 2000b).

Results of both NAEP and TIMMS (Lokan, Ford and Greenwood, 1996) indicate that the difficulties continue into the high-school years. Since the results previously reported by Bragg and Outhred (2000b, 2001) covered Grades 1-5 it was apparent that information was needed for Grade 6 students who had completed their foundational instruction for the measurement curriculum. An investigation of this type is timely as research is beginning to point to the need for teachers to focus on the meaning of the numerals on scales (Clements, 1999), and the conceptualisation of a length as a movement in space (Lakoff and Nunez, 2000) away from a point of origin that becomes zero on a scale (Lehrer, Jaslow and Curtis, 2003).

Methodology

Two studies are reported in this paper. A comparison is made of the results for Grade 5 students in the first study with the results of Grade 6 students using 5 of the tasks from the original study. The Grade 5 data form part of a larger study involving 120 students from Grades 1-5 (aged 6-10 years) from three state primary schools in a medium to low socio-economic area of Sydney. The second study involved 89 Grade 6 students (aged 11-13 years) from a non-selective private girls school in northern Sydney. Following the survey, the Grade 6 students were interviewed in small groups of 5 to 7 to try to ascertain what the students understood about the concepts that were tested. The first researcher interviewed all students towards the end of the school year when they had completed the year's instruction in measurement.

Tasks 1 and 2 were designed to force students to apply their knowledge of rulers in unfamiliar contexts. Task 1 was an 'offset- ruler' question requiring the students to state the length of a shoe printed above a ruler between the 3 and 8 cm unit marks. Task 2 asked the students to measure an 11cm line printed above a ruler without numerals. Tasks 3 and 4 required the students to mark centimetre units on a ruler or centimeter cube. Task 5 asked students to draw what they thought one centimetre would look like if they could see it between the forefinger and thumb drawn on the page.

The tasks have been grouped below to emphasise their conceptual similarities. They were presented in the same order to all students but were not presented in the order seen below.

Table 1 Tasks involving a scale

Task	Description	Knowledge
1	Measure object above a ruler printed between the 3 and 8cm marks.	Length may be measured by counting spaces on a unit scale. A numeric scale can be applied to a congruent set of marks.
2	Measure a line using a ruler with unit markers but no numerals.	As for Task 3.

Table 2 Tasks involving identification of linear units

Task	Description	Knowledge
3	Draw the linear unit on a picture depicting a familiar representation of a centimetre: thumb and forefinger placed 1cm apart.	Identification of the linear unit in a pictorial representation.
4	Identify the linear units on a ruler for a given measure.	Linear units are separated by marks. A numeric scale aligned with marks gives the number of linear units from the origin.
5	State what part of a 1cm cube is used when measuring a length.	The length of an object gives the measurement unit (its area and volume are irrelevant).

Results and Discussion

The results for Task 1 show that the number of students able to state the correct length improved in Grade 6 (69%) compared with the Grade 5 students (50%). Almost a quarter of the Grade 6 students (24%) gave the measure as the numeral aligned with the end of the object. In the small group discussions students were asked why they used this strategy. Ellie's response was typical of the 21 in this group: "You just read the number at the end." Suzie continued, "That's because you can't move the ruler." When challenged to re-measure the shoe with their own ruler, most were able

to see why their original measures were incorrect. Several, however, remained unconvinced, unsure whether to measure from one.

Unlike the Grade 5 students who were more likely to count the spaces, the successful Grade 6 students counted the unit markers from zero. There was also an increase in the number of students who used the most sophisticated strategy of finding the length by subtraction; (12% versus 3%). Few students use this method even though it is used to find a remainder in word problems. Students fail to recognize the 'offset-ruler' task as belonging to the class of 'difference' problems. The length of the object 'offset' on a ruler may be thought of a subset of the length between zero and the end of the object when aligned with a ruler. Classroom tasks explore the concept of additivity, so it would seem to be important for teachers to include tasks that require students to find measures by counting units on scales not aligned with zero. Students need explicit instruction in the use of the 'rename as zero' or the counting of the spaces between unit markers provided that students understand that the measure itself counted linear subunits. A number of students stated that there was "…nothing on the edge of a ruler" and that the numbers "pointed to the lines" [unit markers].

Table 3 Results as percentages correct for Tasks 1 to 5

Task	Name	Grade 5	Grade 6	Comments for Grade 6
1	The Offset Ruler	50%	69%	Measure as the end of the object 24%
2	The Ruler Without Numbers	54%	37%	Count unit marks from one 53%
3	The Finger- Thumb Picture	54%	74%	Draw cube 7%, square 6%, ruler features 11% (e.g. little unit markers or numbers)
4	Units on a Ruler	42%	6%	Indicated unit markers 69%, 21% coloured spaces
5	A Centimetre on a Cube	52%	71%	Indicated square 20%, cube 8%

The results for Task 2 were unexpected. The number of students giving the correct measure fell to 37% in Grade 6, with just over half of this group (53%) counting the unit markers from '1'. This is compared with 54% of the Grade 5 students. In the discussion groups the majority of the students quickly saw that their measure was incorrect when they checked their answers with a ruler. Many of them noted that they should have counted the 'first little mark as zero'. As with Task 1 it seems that the older students were more likely to count unit markers than the spaces between the markers. Esther noted that she "just forgot about the zero". The students were asked why they thought they got confused about where to measure from on a ruler. The majority commented that the measuring they did in the early grades only involved counting from one. Georgia said that the teacher in her previous school told them that "When you use a ruler you don't have to count anything. You just remember to line

up the start with zero and read the number at the other end. Mrs. Z. used to get a red marker pen and draw over the nought so we would remember. I always got it right then."

In Task 2, the unexpected drop in numbers of correct answers in Grade 6 may be due to the fact that greater emphasis is being placed on more complex measurement applications with little time being spent on revision of basic concepts. These results reflect the way students understand the use of a ruler and the cues used for correct alignment. The zero printed at the first unit marker may serve only to direct alignment, while the number corresponding to the end of the object may be seen as the 'measure' rather than the count of linear sub-units. Indeed, some students stated that the number in a measure only told "…where the line ended'. The absence of a zero as a cue therefore elicited a counting-from-one action of the unit markers.

In Task 3, (the Finger-thumb drawing) 74% of the sixth graders (54% in Grade 5) were able to represent a centimetre as a linear unit. It is interesting to note that almost all of the centimetre lines were drawn with a unit marker at either end. The prevalence of these features may be partly due to their prominence on rulers and the emphasis that is placed on them as students learn to mark off the iteration of informal units in early measuring activities. Those who were not successful were almost as likely to draw a square or a cube (13%) as they were to produce a drawing with ruler-like features (11%). The confusion with squares and cubes may have resulted from the use of cubes for area and volume measurement without explicit reference to the feature of the cubes that is used for different measures. The most common ruler-like features were the numbers to 10 or a set of 8-10 tiny unit markers. The belief that the unit marker is the unit of measure was also seen in Task 4 where students had to indicate the centimeters on a ruler. Five students (6%) in Grade 6 correctly drew over the linear units at the edge of the ruler while 61 (68%) marked the unit markers and 19 (21%) coloured in the spaces.

This confusion was also seen in the results for Task 5 (identifying a centimeter on a cube). While the results also show an improvement from Grade 5 (52%) to Grade 6 (71%), it is significant that almost 30% of these students about to enter high school remained confused about the property applied in different measuring contexts where cubes are used. In the small groups almost all the students who drew or coloured squares or cubes referred to the use of the small cubes used in the measurement activities. "Well, it doesn't really make a difference 'cause you just count the cubes" (Sarah, Grade 6). "You don't count the edges unless the question is about 'edges and faces'…Oh, I know, what about the pictures in Signpost, [referring to the student text] we did it in Ms L.'s class. We lined up the shorts and counted them" (Emma, Grade 6). Teaching guidelines state that students should have a clear understanding of the attribute that is being measured (Outhred, Mitchelmore, McPhail and Gould, 2003). If 'shorts' are used for all for length, area and volume, then it is important for teachers to ensure that students understand which feature is counted for each measure.

Conclusion and implications for teaching linear measurement

Research into effective classroom instruction over the last 30 years has yielded important information into fundamental understandings about measurement. The findings reported here confirm that the sequence of early measurement tasks needs to be carefully considered in relation to what students must know about how a scale works and its construction. Of particular relevance is the students' belief that the unit markers are the measures. These markers are, however, at right angles to the length of the object or line being measured and should only be understood as the feature that marks the end of each unit. It would, therefore, seem critical that young students should learn that a length is a linear entity that can be defined. The point of origin can therefore be identified and written as zero. The current practice of filling the space between the endpoints of an object with a line of informal objects called the 'units of measure' may encourage students to consider these objects as they would for exercises in the counting of discrete objects. The continuous nature of measures calls for a counting action that does not use a 'point-count' action but rather a counting action reflecting the essential movement from the point of origin. This is only possible if students are able to see or visualise the linear units and that the count is made by moving a finger for example, along the units counting as they reach the end of the unit. The purpose of the unit marker may be more easily seen as the point where each unit starts and ends. The unit markers on rulers are particularly prominent features that, in the absence of careful instruction about their function, may become the focus of a student's counting.

A line used to represent a length also satisfies the requirement that students should learn to identify the attribute (Outhred et al. 2003) and helps students discriminate it from the units used to measure length and volume. The consequence of using a concrete representation of a length is that the unit of measure will be defined by the length of object or action chosen as the informal unit. A line of objects, for example paper clips, allows the students to draw the unit markers and when the clips are removed the linear units thus created are counted from zero not the unit markers or the informal units. An analogous relationship is created that has a closer fit to the form of a scale. This relationship may also be satisfied when students iterate a single unit or use an action such as hand spans. Early measuring activities should emphasise the measuring process as having three steps: 1) Defining the length and assigning zero, 2) iterating the unit and 3) counting the linear units created. Teachers should also be aware that counting skills and strategies used to find a measure should be introduced and discussed with the students. Since a correct measure can be found by counting the spaces or unit markers it is critical that students understand what the measure obtained actually counts.

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In the Serpent's Den: Contrasting Scripts relating to Fear of Mathematics

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The author reports on data taken from a single case study of a mathsphobic student teacher engaging with a Mathematics content and method course as part of her pre-service course in primary teaching. Sample comments are given from the journals of both student and lecturer as the course progresses. The interpretation of this data is then explored from a variety of perspectives in order to begin to untangle the complex web of factors, which interact with each other in this topic.

INTRODUCTION

> For twelve years I watched it writhe upon my page; fearful of its poison – that feeling of hopelessness, which it can so easily induce.
>
> *'Maths is everything'* *'Without Maths you are nothing"*
>
> I passed. Every year I passed. At the end of school I derived much pleasure from the belief that I would never again have to bow down to Maths – the conqueror serpent. My delusion has since been shattered. I must now face Maths again. (Pegg 2001)

The above quotation comes from the journal of a student taking the one year pre-service Post-graduate Certificate in Education (PGCE) (Primary) at the University of Cape Town. Attaining this Certificate allows students to become a generalist teacher in South African primary schools. This means that one of the compulsory courses, which all students have to take, is a year's module on the Content and Method of Primary Mathematics. In a previous PME report (Breen 2000), I explored the occurrence of strong emotions of fear and anxiety towards mathematics experienced by these mature students. Later work (Breen 2001) has seen the development of a curriculum for the first semester of this course which attempts to address the mathematics anxiety found in the class and draws strongly from an enactivist approach to cognition (Davis 1996; Varela, Thompson and Rosch 1991). Carroll (1994) explores the occurrence of mathsphobia in a similar group of students training to become teachers. Her case study follows the earlier work of Buxton (1981) and Frankenstein (1989) in attempting to understand how mathematical achievement can became blocked by emotional and sociological factors.

THIS STUDY

As suggested by the quotation at the start of this report, there are students in the above PGCE class each year that, right from the start, exhibit signs of a powerfully negative relationship with mathematics. Since it would be unethical to simply ignore and then just fail such students without engaging with them, this research report will focus on one such mathsphobic student, Marissa, who was registered in the same year as the author of the original 'serpent' quote.

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The Content and Method of Mathematics module is structured in such a way that I teach the first semester's work by means of a double lecture (90 minutes) twice a week for 12 weeks. Data for this case study was obtained from the journal that the student kept for the class as well as from her end of year Research Essay and Teaching Practice journals. Further data was obtained from recorded interview material of our end-of-year revision sessions and my own teaching journal for the class.

The data will be presented in the form of a time line of extracts from what I consider to be significant moments or utterances as recorded during the interaction. Then intention of these extracts is not to present the 'truth' of the event (as both lecturer and researcher I believe this would in any case be an impossible task), but rather to present the reader with as rich a picture as possible of an account of Marissa's engagement with the mathematics course and the lecturer.

THE DATA

First Term

Marissa arrives: First session (27 Feb).

In this first session of the year, each student is asked to come up to the board to write down their hopes and fears for the year. Shortly afterwards, Mr. Smith arrives in an academic gown and gives them a test. He supervises this test in an unpleasant way by shouting at those who make mistakes and putting pressure on the students in various familiar ways.

> I notice a small woman sitting near the front of the class chewing away frantically and blowing gum. She could be trying to draw attention to herself. Is she going to be one of those cocky students who think that they are wasting their time in this class because they can already do the maths? She doesn't say anything but I sense that this is not the case. At the end of the session I test the latter view out by saying to her that it seemed to me that she was chewing her gum furiously to try to ease her tension. She agrees with me. (Chris: Feb 27)

> *When I first realised Maths was inclusive in this course I immediately felt physically ill. I can honestly say I dreaded the first lesson. Maths since school for me has always been a stumbling block. I always felt inferior to those who were able to do it. I (and it is my own insecurity) always felt that people thought less of me that I did not do Maths. (Marissa: Feb 27)*

Second Session: Matchsticks (2 March).

In this session the students use matchsticks as a physical bodily activity to make the patterns from which they will generate formulae through visualization. My focus as teacher is to encourage all answers as being different ways of 'seeing' the pattern, and to discourage shouting out and competition as a way of encouraging diversity.

> *Fear and anxiety engulfed my entire body. The prospect of actually doing anything mathematical was beyond my realm of consideration. Even before*

we began I said to myself I probably won't be able to do it so why bother? These words are remnants of 'my life with Maths' at school – my motto was I can't do it so don't even try! The matchstick exercise forced me to engage with Maths. The first answer I knew I would get wrong and did! So I thought here we go again – the idiot of the class. But when I did eventually work out one of the patterns I began to physically relax my tense body and mentally relax my mind. Chris, you seriously make me feel at ease, I am not saying this to 'score points' but I feel almost "safe" with you in the class. It is though you act as a 'buffer zone' between me and the rest of the class – who I feel are too confident in their ability in Maths. (Marissa: 2 March).

Fifth Session: Painted Cubes (9 March)

The students have been working on getting formula for generalizing patterns for 6 hours by now, so I push the pace in this session. They are put into groups and given apparatus and asked to solve a multi-leveled problem.

Marissa comes to me and says that she can't cope with this problem and starts crying. She says she is going to vomit. I say that she can go outside to the toilets and vomit or she can use the bin and stay inside but she is not allowed to use the vomiting as an excuse to stop doing the maths. My recollection is that she left the room and then came back in. I suggested that she might like to work on her own and she could then ask me questions. (Chris: 9 March)

Well the 'after shock' of Fridays lesson did not go away that easily. I hated myself for allowing myself to be so vulnerable and immature firstly in front of the class and because I could not do the work! From the moment you put the blocks in front of me I could immediately feel the anxiety building up. As I told you I really did feel physically sick to the point where I felt as if I could not breathe. You knew I wanted to give up and bolt from the class, I wished I could! To make matters worse I felt the pressure of everyone around me, their confidence to tackle the problem made me feel unworthy and stupid. Although I still felt uneasy when I went to sit by myself I really did try to calm down but as soon as I got stuck, panic seized up again…Chris, I seriously want to thank you from the bottom of my heart for your help and encouragement on Friday! It really helped me to know that you were on my side. (Marissa: 9 March)

April Teaching Practice (TP1).

At the start of the second term, students are placed in schools for a five-week period to teach and have their teaching assessed.

Today I had the opportunity to participate in a double Maths lesson, which I ran by myself with a little assistance from the teacher whom I had asked to be present. At this point I have to explain what an "achievement" this was for me even to enter a Maths class let along as a teacher! It was ironic

because the class was busy with equivalent fractions and before T.P.1, we were busy with equivalent fractions in our Maths method class. Therefore I used the same idea that was used for our lesson. The idea proved to be successful and I think the learners really enjoyed working from practical experience. I would see and relate to how their faces lit up when they got the answer right. (Marissa: April TP Journal)

The Mock Test. 25th May

At the start of the term I give the class a content test, which covers the topics of basic operations, fractions, ratio and proportion and percentages, which will be covered in the second term. My aim is to check the level of the class, but also as a way of identifying concepts that need to be addressed in the sessions.

> Marissa left the room early in tears and stayed in the ladies toilet for a long time. I asked one of my female colleagues (J) to go in and talk to her. Marissa ended up spending a lot of time in J's room. J later told me that M's maths thing is serving as a "handle" to hold a lot of fears about the whole year in place. (Chris: 25 May)

Marissa gets the fewest sums correct in the whole class. She omits all questions containing division and fractions. She failed to attempt an answer to the questions: 4,9 ÷ 0,007 and 63000 ÷ 210.

Content sessions during the rest of the second term.

Every Maths lesson comes with its new challenges for me, with regard to today's lesson I was obviously quite tense, knowing the outcome of my disastrous mock test and that I basically had to 'relearn' many concepts. I felt overwhelmed at this challenge. (Marissa: 30 May)

I still felt a bit 'edgy' with grasping the concepts due to the fact that basically this was 'new' to me. I felt that everyone else in the class 'clicked' on instantly while sometimes I didn't. (Marissa: 1 June)

Today I must admit was on the one hand relief, to be finished with all the content, but on the other hand, knowing that I still have to write the test…I really will try my best Chris. (Marissa: 13 June)

The first test proper. 20th June.

Marissa completed the basic arithmetic content test without having to leave the room early and gets a total of 25 marks out of 75. She was able to answer the question 19440 ÷ 72 correctly, but attempted to answer the question 0,95885 ÷ 0,0245 by placing the two numbers in columns and then going no further.

Revision Sessions (25 Oct – 13 Nov):

In the lead up to the final test at the end of the year, Marissa asked if I would help her with her revision. I agreed on condition that I could tape these sessions and also that I could interview her about some aspects of the year as well as her past experiences

with mathematics. The following extracts are taken from the last session before the final test where we are going over a problem she is tackling on division:

Extract One:

C: 6 divided by 3 goes like that, which number is first
M: this one – the 6
C: where does it go?
M: inside, that's what I thought
C: write it out in full, okay and outside – okay, so there we can do it, that's how it goes, because you go 6 divided 3, you can write that one down, that's the first step, not that, you couldn't do it here

Extract Two:

M: I know but I'm saying: Do I do this into this whole number or into that first?
C: Ja, just long division
M: 165 into 429 – it won't even go!
C: Would you just do it please? You get sulky and grumpy, that's what you do, and then you suddenly lose it. Just hang on, just hang on. Here this is a whole lot of rubbish because you didn't go for that as I told you.

The final test: 14th November

Marissa comes to my room to write the compulsory test. She is the only student in the class who has not obtained the sub minimum requirement for this topic. She correctly answers the first division problem of 42,90 ÷ 16,5 and manages the decimal comma for the first time in a test. For the second problem, 0,6 ÷ 0,0012 she gets the divisor the correct way around and has written 6 ÷ 2 in the margin (see above extract from interview). However, she reaches an incorrect answer of 0,05.

> *I would've given up, I really would have, I would've left it blank, if you weren't here, I would've gone, stuff this, big deal, can't do it, move on, and left it out probably. I am actually pleased that I've done it, I really am because I mean I can do it, you've just shown me that I can do it, but it has been tough mentally.* *(Marissa: post-test interview)*

INTERPRETING THE ACCOUNT

Lather (1991, 91) identifies an epistemological shift in research methodology away from an emphasis on general theorizing to problems of interpretation and description. She argues that description/interpretation inevitably involves bringing to the fore one's own perspectivity, which presents a challenge to conventional views of objectivity. It has already been accepted that, since only a portion of the data has been selected for presentation here, this must already represent a subjective choice. But what of the analysis of the account which has been presented?

Marissa's script as to what happened.

I felt that I was incapable of learning any Maths concepts and my expectation of failure induced the behaviour that increased the likelihood of that outcome. I believe my failure in Maths was as a direct result of my negative self-expectancy (Research Essay). I think it all started in Form 4 when I couldn't understand a thing about

division (13 November interview). Indirectly my negative attitude towards Maths can be attributed to the fact that both my parents also believed that they weren't good at Maths and would often comment that "our family is just not good at Maths (Research Essay). This self-fulfilling prophecy was the legacy I brought with me into my Maths method course this year (Research Essay)

It helped such a lot this year to have a teacher who believed in me and that I could do the mathematics. I liked the way he asked us to visualize things with apparatus. Talking it through with him always helps as he has a calming effect on me (13 November interview). You can see that it worked because my marks improved so much during the year. I almost completely conquered my difficulties with division.

Chris asked us to keep a journal to record what we felt like after each lesson, I found this to be therapeutic in that it was the first time in my life that I was able to express my total anxiety with regards to Maths and to know that someone cared whether I passed or failed (Research Essay October 2001). The experience on teaching practice was the most significant event. I do not think anyone can realise what a personal triumph this was for me. I cannot believe that I did not panic or end up crying in the lesson. Although my anxiety towards Maths still exists I REALISED THAT I CAN DO IT! I felt as though years of suppressed anxiety had literally been lifted off my shoulders (Research Essay).

Chris's script: Version 1

It seems that I'm really getting somewhere with this course on two levels. In the first instance, the focus on enactive principles has allowed Marissa to voice her feelings in a reasonably 'safe' environment. She has been forced to stand back a bit from the events of the day in the class by the set task of keeping a reflective journal which forms part of the assessment portfolio. This work is really important and it showed when she plucked up the courage to teach maths on TP and it worked for her! It also proved to be important to be strict with her at times as shown, for example, in the incident where she wanted to revert to her school strategy of leaving the room for the rest of the class with a headache. Even in the final revision session, being firm with her allowed her to bring the taught example of using $6 \div 3$ as a reference point for understanding how to order the numbers in a division sum. The gradual improvement in her ability to tackle division problems during the year was also encouraging, especially as this was the very same topic that had started off her whole path of negativity towards mathematics. Marissa also showed me the important lesson of always believing in a student no matter how much they might challenge you.

WAIT A MINUTE…

The above interpretations mirror claims from a research report on an intervention programme with a similar aim in which there were resulting 'genuine and sincere affective changes' (Grootenboer 2003, 419). However research such as that by Evans (2000) and Blanchard-Laville (1992) provide an opening for an alternative and less comfortable interpretation of events, which might read as follows.

Chris's script: Version 2.

Over the years I have developed a strategy to get students who struggle with maths to talk to me. It's based on the good cop, bad cop routine. In the very first session I introduce them to a nightmare teacher called Mr. Smith who draws out memories of bad times in their school maths classrooms. At the end of the role-play I remove the symbol for my transformation to Mr. Smith (an academic gown), and return to the class as the good guy, Chris. The acceptance of feelings and the use of apparatus and visualization as ways of getting into the mathematics, helps distance both my class and me from Mr. Smith and his world. It also promises the students the possibility of a different mathematics experience.

Marissa's mathsphobia basically stems from a mathematics script written many years ago. In fact, it is probable that most of it was written in her very early years before she encountered school mathematics. This powerful condensation of mathematics is considered by many to be a common heritage for all students (Tahta 2002).

In this particular class, I am both a teacher and a teacher educator, but not a psychologist so I cannot engage with this early script. My task then is to play along with the idea that this fear has everything to do with mathematics. I tackle this problem with mathematics as a short term problem needing patching up as quickly as possible. However, I also aim to leave open the possibility for Marissa to choose to work on whatever else comes up - in her own way. In Marissa's case my strategy of positioning myself as a good guy works and she comes to view me as someone who at last has belief in her. I listen to and read about our interactions in class and in her journal writings. I try to pick up her cues as to when to be firm and when to give her space. Even on the day before the final test she still does not understand division, so I am forced to take her on one of those meaningless step-by-step routines where she ends up with correct answer because I have been firm and she has been anxious to please. Fortunately, this lesson stayed with her (almost) for 24 hours as is shown by the improvement in her performance in division problems. However I choose not to dwell on the fact that she is unlikely to still be able to do the problem next week.

Cabral and Baldino (2002) discuss the concept of pedagogical transfer where positive transfer is identified with love. However, they warn that this 'love is to be distrusted, since the student is only seeking the way to produce the right answer, so that …(s)he will be recognized as one who knows'. Marissa ends the examination session by checking whether I will be in my office the next day, as she wants to bring me a gift of appreciation. I'm relieved at the end of the year that it has all worked out, even though at times it became increasingly difficult to be supportive. She has had a good teaching practice experience and she has improved from a mark of around 10% to one of about 60%, and has passed the year. However, a year later, she has still not arrived with the present! Is this proof that the love is to be distrusted?

CONCLUSION

The mathematics anxiety that student teachers bring to their training as primary mathematics teachers is a serious and complex issue with many levels. This report

has attempted to provide a window for the reader to enter into the lived experience of one such student as she engages with a mathematics class which has been specially designed for students with similar difficulties. The provision of an alternative interpretation of the account has attempted to begin to unwrap one additional possible layer of the complexity. Readers are invited to attempt to continue this process by providing their own interpretation of Marissa's story.

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FORMAL INCLUSION AND REAL DIVERSITY IN AN ENGINEERING PROGRAM OF A NEW PUBLIC UNIVERSITY

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Political measures are being taken to "democratize" access to universities in Brazil. A new State university recently created after a wide consultation to the population, has taken two important measures: (i) it reserves fifty per cent of its places to poor students and ten per cent to physically handicapped ones and (ii) has abolished the departmental structure. The paper discusses the result of a strategy adopted to deal with the highly heterogeneous classes resulting from the first measure and presents a way of taking advantage of the absence of epistemological control by a mathematics department to offer interdisciplinary objects as possible students' objects of desire.

INTRODUCTION

"I, Diogo Isidoro Gurgel Mascarenhas, declare that, among other possessions, am the legitimate master and owner of a slave of the name of Ana received form the heritage of my father, Lucio Gurgel Mascarenhas and since the said slave is my mother, and since today I come to age, as certified by the marriage of yesterday, therefore, finding myself in the right, I concede to the said my mother full freedom that I concede with all my heart" [Coimbra, 2003].

It seems that only recently Brazilian society has realized the severe exclusion processes, racial as well as economical, that are still going on since much before 1869, date of the above public statement. A feeling for inclusion is now sweeping the country and measures are being taken, such as the "zero-hunger" Presidential program and the "democratization" of access to universities. However, if inclusion becomes necessary it is because diversity has already made ravage. A new public university[1], recently created after a wide consultation to the population, has taken two important measures towards inclusion: (i) it reserves fifty per cent of its places to poor students and ten per cent to physically handicapped ones; (ii) it has abolished the departmental structure. The first measure is directed against entrance discrimination and is justified by the argument that poor students deserve a *compensation* for being confined to public schools, held as offering a lower quality teaching than the expensive private ones. The second measure is directed against the exclusion process inside the university under the argument that the hard-sciences departments crate themselves into narrow epistemological conceptions, develop a

[1] UERGS, State University of Rio Grande do Sul, the southernmost State of Brazil, was created in 2002 as a multi-campi university devoted to boost culture and production throughout the State.

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sneering attitude towards other areas and are responsible for the failure of many students driven out of the university.

The paper reports on a research project to deal with the conditions that these measures impose on the mathematics classrooms of an ambitious Engineering Program on Digital Systems in the periphery of a three-million inhabitants urban agglomeration in south Brazil. Students of the traditional federal down town university prejudicially refer to us as "a university for deficients". We engaged in this Program in August 2002 and were entrusted with the teaching of six one-semester courses on calculus, analytical geometry and differential equations. We immediately established a research project in Mathematics Education that includes our teaching activities and is guided by the question: how can mathematics teaching best supply the demand of this particular engineering program? The degree of generality of our findings hinge on three points that may be common to other universities and programs. 1) The reservation of places obliged us to deal with highly heterogeneous classrooms in an effort to revert social exclusion through mathematics [Gates, Lerman, Zevenbergern, 2003] and having "social justice as a desirable outcome" [Mesa & Sounders, 2003]. 2) The absence of a mathematics department set us free from the control of the mathematical science and obliged us to seek the legitimacy [Chevallard, 1989:63] of our objects of teaching in the consensus of colleagues of subsequent professional courses, specially the physic teacher, seeking to "integrate the teaching of mathematics and sciences" [Keller and Marrongelle, 2003]. 3) Since the University is new, there is no tradition that we can count upon nor that can hinder us. On the contrary, the contribution of the institution to the formation of systems of beliefs [Gates, 2001] leading to the formation os students identity [Brown, 2003] as future engineers is to be established by us, in so far as we develop our activities.

METHODOLOGICAL AND THEORETICAL CONSIDERATIONS

The need to impress change on reality and to build tradition, led us naturally to adopt the methodology of action-research. The colleagues to whom we communicated our research intentions agreed that these were relevant for the Institution and agreed in constituting the research forum of such a collective problem.

The main proponents of action research [Elliot, 1991, Zeichner, 1998] seem not to have been able to avoid action research to be considered as a second-class research method [Cohen and Manion, 1994:189-193]. They do not clearly dismiss the idea that the necessary reflection accompanying any teaching action aiming at enhancing learning, might be taken as genuine action-research. Therefore "some teachers see their practice of planning, teaching and reflecting on teaching as a research process" [Jaworski, 2003]. Although we too carry out such reflections, they are only part of our research problem, which is adjusting mathematics teaching and creating tradition to a new Engineering Program. Accordingly we adopt a wider conception of action-research:

> "Action research is a kind of empirically based social research that is conceived and carried out in close association with an action or solution of a collective problem in

which the researchers and the participants implied in the situation or problem are involved in a participative or cooperative way" [Thiollent, 1998:14].

Furthermore, we contend that there is no research that does not impress some change on reality and therefore *is* action research, *lato sensu*. However, due to a narrow concept of 'action' many researchers ignore or avoid to consider the social effects as part of their research's action. Once attributes have been assigned to different institutionally defined social agents as 'the teachers' and 'the researchers', the unavoidable consequence is a problematic relation between theory and practice and an increasing hierarchy between academy and school. The results are the many not very fruitful efforts "to bridge the traditional gap between theory and practice" [English, 2003], "to bridge the school/university divide" [Jaworski 2003], to solve the "dilemma" through "collaborative research" [Carrillo, 2003].

Once research extends its focus to include its own actions in reality, human subjects including the researchers, are implied and the field of affect is open, together with cognition. J. Falcão suggests that the dichotomy affect/cognition can be avoided if we choose "a more productive unit of analysis (that) targets cultural situations in the context of which a mathematical activity takes place involving a set of identifiable epistemic contents (a conceptual field)" [Falcão et al, 2003:274, Falcão, 2003]. However, if we accept to "look at social forces not only as acting *on* us but also as acting *in* us" [Gates 2001:18] we are led to recognize our own state of dependence as affective human subjects and the hope for an external point of view from which a synthesis of affect and cognition would be possible, vanishes. Thereafter a theory that places at its very center the dialectics of the Subject and the Other, a *mismatch* as constitutive of the Subject, becomes necessary and we are led to the philosophy of Hegel-Lacan as a theoretical framework.

According to Lacan, the theorizer has to accept that no theory will never cover up reality and the research-teacher will have to fully assume his/her part in the interplay of desires in the institution, including the classroom. However, since Adam left Eden, desire is not substantive, it lies only in its effects and presents itself through *objects of desire*, none of which is perfectly satisfactory, so that actions do not generally match declared intentions and the game has to restart. To make the mathematical object into his/her students' object of desire is the dream of all mathematics teachers. We are a little more ambitious, we seek to constitute *institutional professional objects* with mathematics built-in, as students' object of desire. To realize our dream we count on our ability to work out the *pedagogical transference* [Cabral and Baldino 2002] starting from our declared intentions as research-teachers. Our intentions hinge on three conceptual heads: epistemology, didactics and pedagogy.

EPISTEMOLOGY

By *epistemology* we mean the institutional legitimacy of the objects of knowledge brought into the classroom. In accordance with our colleagues we decided to make wide use of infinitesimals, not as a metaphor or an *ad hoc* strategy to find integral

formulas, but as a true *object of teaching*. In the exams the students are required to calculate differentials of polynomials and rational functions by the *method of infinitesimals* and to write down the *infinitesimal elements* of area, volume, pressure, moment of inertia, electric field and so on, *before* they put an integral sing in front of them and choose the limits of integration. The number line (the *continuum*) is referred to as being "thick", that is, as holding infinitesimals, monads and infinite numbers, besides real numbers, just as we contend that Cauchy thought of it [Sad, Teixeira, Baldino 2001].

The epistemological difference between infinitesimals and limits about, for instance, the concept of area is striking. According to the Weierstrassian theory of the "meager" real continuum the area is *defined exactly* as a certain real number obtained from Riemann sums. The student is asked either to reformulate his previous concept of area so as to adjust the definition or, at least, never to refer to it as 'area', because, thereafter, this word will have a new 'precise' meaning. Mathematics becomes a formal science. On the other hand, if we adopt the "thick" hyper-real or Cauchy's continuum, it is the concept of area that develops itself, both historically, from the Egyptian scribes to the present, as well as logically, from the simplest Brazilian peasant to calculus textbooks. At a certain point of its development, the concept of area *expresses exactly* the area under a curve as an infinite sum of infinitesimal elements $\int_a^b f(x)\,dx$ and this area is calculated *approximately* (up to an infinitesimal) as $F(b) - F(a)$. Mathematics remains a conceptual science. We clearly aim at establishing the infinitesimal way of thinking as a tradition of the Program.

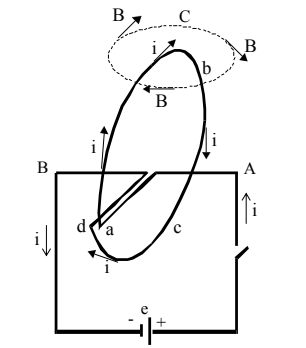
DIDACTICS

By *didactics* we mean the ordered set of mathematical objects introduced into the classroom as a focus for transference. It was decided to introduce objects from other courses into the mathematics classroom whenever possible. For instance, we introduced the RLC series circuit as a genuine object for our differential equations course, hoping to make it into an institutional professional object, since it is also studied in physics and forms the basis for electrical circuit courses. In the action-research forum it was decided that we should not work the associate second order differential equation in terms of the electric charge in the capacitor, as most mathematics books do, but adopt the integro-differential form, in terms of current, as needed in electricity. We reviewed our undergraduate physics courses so as to be able to call for students' understanding of physical phenomena and to use the proper units and meaning of constants. The following episode is illustrative of such an attempt.

We had set the comparison of the integral and the differential forms of Maxwell's equations as the main objective of our third-semester two-months course on vector calculus. We started the subsequent two-months course on differential equations with the harmonic oscillator as a model for the mass-spring system and the RLC circuit. So it became natural to ask the students to explain the performance of the inductance from the Maxwell's equations. In searching the answer to our question we stumbled

into a paradox that kept us puzzled for a couple of weeks. We introduced the paradox to the students, told them that we did not know the answer, suggested that they asked other teachers in ours and in the federal university that they might contact and, promised them a bonus for intelligent suggestions. We received many interesting contributions from experienced colleagues, both in mathematics and in physics, before we could find out our mistake that had also passed unnoticed to them. We contend that the paradox has arisen because of our insistence in integrating physics and mathematics in the same object. Indeed in most physics manuals, Maxwell's equations are presented at the end of the course, after the Laws of Faraday for induction have been discussed, so that explaining inductance from those equations is out of question. The exception is Feynman [1972]. This paradox worked as a genuine institutional object for two weeks. It could only arise because the dichotomy of different departments was surpassed by the university inclusion policy. Here is the paradox; due of lack of space we leave its solution to the reader.

Consider a circuit with a coil and an *emf e* (figure). For the qualitative analysis made here, a one-turn coil is as good as an n-turn one. Turning the switch on, an increasing electrical current starts circulating in the indicated sense. Hence *the potential in A is greater than the potential in B*.



A paradox

The fourth Maxwell equation applied to the curve *C* (upper part of the figure) implies that the circulation of the magnetic field around *C* is proportional to the current *i* across the surface bounded by *C* (the parcel due to the variation of the electric field across this surface being null). Since the current is increasing, the magnetic flux across the surface *abcd* bounded by the coil is also increasing. Considering the curve *abcd* that follows the turn of the coil clockwise, the third Maxwell equation implies that the circulation of the electric field along this curve is *minus* the rate of change of the magnetic flux across the surface *abcd*, being therefore negative. Hence the electric field along this curve must be pointing in the opposite direction with respect to which the circulation is calculated, that is, in the sense *dcba*. But the electric field always points from the higher to the lower potential, hence *the potential in B* (the same as in *d*) *is greater than the potential in A* (the same as in *a*).

PEDAGOGY

By *pedagogy* we mean the set of rules, the work contract that regulates the relation student/object; these rules lead to the constitution of institutional professional objects. The first negotiation of the work contract was very difficult. There was no 'last semester' to which we could refer in order to introduce our classroom rules that included mandatory daily-assessed group-work with teachers' expositions at the end. A considerable group of students (group A) voted for traditional lectures. Within two

weeks most of them stayed in the garden, playing cards during the expositions that they had asked for. Recently one of their leaders confessed that she always passed with good grades in elementary and high school but never had to open a book at home. Now when she finally convinced herself that there was no other way to get a passing grade she was facing difficulty in getting concentrated. Some of these students are of the kind that we would call 'bright'. They always have a ready answer to any question, no matter how complex it might be. Most of their answers do not make sense or make only an allusive sense. Typical of their discourse is the following pearl: 'the gradient is the slope of the only tangent line to the tangent plane'.

Other students of group A do not display the same brightness; nevertheless they try to behave as if they did and they join the others in the game of cards. Strange behavior of people who travel seventy kilometers everyday back and forth to come to the university and say that they want to master high technology, we thought. Where is their desire? We soon found out.

Indeed, in the first few days of course we came upon a group of students (let us call it group B) who did not recognize the function x^2 and could neither produce its table nor draw its graph from the table. Here is a true mathematics educational problem, we thought, and we set ourselves to solve it. This group represented one third to one fourth of the students of each of our two forty-students classes. We thought that if we succeed in making this group to show a reasonable progress, all the rest would follow. We committed ourselves to "leave no student behind", we arranged our class schedule so that both could be always present in the classroom and organized extra classes for these students. We put one student at a time at the black-board and asked for contributions from the others as described in [Cabral and Baldino, 2002]. We soon found out that their difficulties were bigger than we thought. These students could not use the rule of three, when asked how much is twelve times 147 divided by 147 they required a calculator, they could not evaluate the simplest arithmetic expressions nor solve the simplest algebraic equations and for them, the largest side of any triangle was always the 'hypotenuse'. One of them could not solve this problem: four chickens weigh five kilos; how much weigh two chickens? After half an hour, under our insistent stimulus, her answer was 2,500 grams. Proficiency in our mother language was of no help either: at least two of them, after several attempts, could not repeat the statement of Pythagorean theorem without reading it, much less make any sense of it. During regular classes we dedicated special attention to these students. Sometimes one of us stayed among them while the other took care of the rest of the class. Nevertheless, they could not get started in the exercises of the day [Stewart, 2001 bended by work-sheets] because they could not make sense of what they read. In fact, each student in group B would require a full time tutor during the whole class. They asked their colleagues for tutoring but soon found out that these were not patient enough to deal with their difficulties.

A few of these students showed some progress during certain classes but in the next day it was as if they had never lived through the previous one. The most dedicated

ones had an astonishing ability to produce the answers that we expected, thereby producing an illusion of understanding. We carefully avoided such a trap and led the transference relation strictly toward the mathematical objects. In spite of all efforts group B failed all the exams and along the three subsequent semesters gradually quit the Program.

In spite of our efforts, at the closing of the third semester (Dec. 2003) the 80 students admitted in August 2002 exhibit the following situation in mathematics courses: 5 out of 32 (15%) who were admitted into reserved places are getting passing grades and 27 either quit the course or are failing repeatedly; out of a total of 23 who are showing some success, 27 (78%) were admitted into regular places. The situation of the 40 students admitted in March 2003 point to the same tendency. All the thirteen students composing group B were in reserved places.

The behavior of group A now becomes comprehensible. This university was created around a strong feeling for inclusion and it was estimated by the research forum that rejection of a large number of freshmen would certainly raise disapproval of the upper administration. We are all hired on a two-year temporary basis. Group A certainly realized our constraints and estimated that, as long as the not negligible group B was present and had any chance of passing, as we expected, they would certainly pass too, without further effort than coming to the university just to pick up some indications during classes. The cards game was natural, in spite of their declared intentions of becoming professionals capable of mastering high technology. Only at the closing of the third semester some signs of change are being noticed. Out of the 80 students entering in March 2002, 20 concluded the third semester mathematics course and 16 passed. A Hard-working group of able students was finally formed but through a high social price.

As students of group B abandoned the Program, they expressed strong feelings of revolt and blamed "our method" for their failure. 'We expected new teaching strategies but what we saw was exercises from a book', they said. They never declared it explicitly but we suspect that they would hope for the widespread "method" used in high schools: one solved sample exercise followed by several equal ones and one of them chosen for the exam. We understand their distress: like the personage Jim in Spielberg's Empire of the Sun we had believed that we had the power to restore their mathematical ability and we had unwillingly transmitted them our belief.

A FINAL WORD

Compensatory policies have certainly become necessary in face of the severe social processes of segregation in all areas, not only in Brazil. However, the indiscriminate reservation of places in universities without further consideration about the problems that this decision raises, reduces the whole policy to mere demagogy with perverse social effects.

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PRIMARY STUDENTS' UNDERSTANDING OF TESSELLATION: AN INITIAL EXPLORATION

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Tessellation is included in many mathematics curricula as one way of developing spatial ideas. If students do not understand tessellation in the intended ways, however, the development of other spatial ideas, such as properties of shapes and symmetry, may be compromised. Van Hiele levels were used as a basis for analysing the descriptions of eight different tessellation patterns by 26 Year 5 and 6 students. Most children saw the underlying structure in terms of 2D shape. Responses from some students indicated that they understood the tessellations only as movements of shapes or saw many of the patterns in 3D. The implications of these findings for teaching are discussed.

INTRODUCTION

Although tessellation has mathematical applications in fields as diverse as biology, architecture and physics, in the school mathematics curriculum the topic tends to be included as a means of developing students' understanding of geometrical ideas, rather than as a worthwhile mathematical idea in its own right (e.g. Australian Education Council (AEC), 1990; National Council of Teachers of Mathematics (NCTM), 2000). In the Australian mathematics profiles (AEC & Curriculum Corporation, 1994) tessellations are explicitly mentioned as indicators of space outcomes in the middle years of schooling, although the progression from one level of understanding to the next is not clearly defined. There appears little attempt in curriculum documents to describe a coherent developmental sequence of understanding of tessellation. If tessellation is a topic included in order to develop students' understanding of shape it would seem desirable to be able to describe children's appreciation of tessellation itself, since misconceptions of this topic could affect the development of other spatial ideas.

Owens and Outhred (1997) described young children's difficulty with visualising tiling patterns, particularly when the shape to be tiled was unfamiliar. The focus of this study was the concept of area, and a large proportion of the children in this study were unable to quantify the number of tiles needed to cover a particular shape. Vincent (2003) suggested that exploring tessellations was one approach to investigating the properties of 2D shapes. Serra (1993) included tessellation explorations as one way of developing ideas about symmetry. The notion of using tessellation activities as a means of developing understanding of other aspects of geometry may be compromised if students do not understand the nature of tessellation, or have different perceptions from those assumed by teachers.

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One approach to considering students' understanding of geometric concepts is that provided by the Van Hiele levels (Burger & Shaughnessy, 1986). The five levels, visualisation, analysis, abstraction, deduction and rigour, are hierarchical in nature, and describe increasingly complex reasoning about geometry. The key aspects of each level are:

Level 0 (Visualisation): Students take account only of the appearance of the shape, and describe properties only in terms of its appearance.

Level 1 (Analysis): Students describe properties informally, and can establish the essential conditions through a consideration of component parts.

Level 2 (Abstraction): Students can draw on logic to establish necessary and sufficient conditions when describing the properties of shapes.

Level 3 (Deduction): Using formal reasoning systems, students can establish theorems and rely on proof as the ultimate authority.

Level 4 (Rigour): Students can move outside a single system and compare and contrast geometries that are based on different premises.

The van Hiele levels have been primarily applied to understanding of the properties of 2D figures. Where tessellations were concerned, the understanding demonstrated needed to take account not only of the shapes involved in the tessellation, but also of the transformations applied in order to create the tessellations. At Level 0, students could be expected to identify the shapes involved in the tessellation or the combination of shapes that made up the basic unit that was transformed to create the tiling pattern. Students responding at Level 1 would be expected to describe both the shapes involved and the movements used to transform the shapes but in a disconnected fashion. Those responding at Level 2 could be expected to integrate their descriptions, using some level of technical language such as "flip" or "slide" which is commonly used in primary classrooms. In a very good Level 2 response, there might also be some attempt to quantify the extent of the transformation. Since the students involved in this study were in Years 5 and 6, at the end of primary school, it was not expected that the higher levels of deduction and rigour would be observed.

RESEARCH QUESTIONS

The research questions for the study were:

1. How do primary school students describe tessellation patterns?
2. Are the van Hiele levels useful in describing students' understanding of tessellation?

METHOD

Teachers who participated in a professional development program, Success in Numeracy Education (SINE) that targeted the middle years of schooling (Years 5 to 8), as part of the program were provided with a set of assessment tasks (Callingham,

Clarke, Falle, Inglis, Lawrie & Pegg, 2003) that addressed different aspects of the Space strand of the mathematics curriculum. All teachers agreed to trial at least one task with their classes, including the tessellation task of interest in this study.

The tessellation task was designed to be used by a whole class, but not necessarily under standardised conditions. The task was exploratory in nature, with the aim of providing formative information to teachers so that they could more effectively plan their teaching programs.

Eight different tessellations were presented in pictorial form, and students were asked to identify the shapes used and then to describe in as much detail as possible how the shapes were transformed to create the tessellation. The tessellations used are shown in reduced form in Figure 1. In all instances the responses presented were written, although teachers were allowed to scribe for their students where appropriate. The tessellations included regular and semi-regular patterns, one non-periodic tessellation, and patterns that included pentagons. It was anticipated that many of these patterns would be unfamiliar to students, although they were expected to recognise the underlying shapes that made up the unit of tessellation.

Using van Hiele levels as a basis, a set of descriptors was developed to provide a basis for the analysis. These are shown in Table 1. No descriptors were provided for the higher van Hiele levels, although it would be possible to describe these, including aspects such as quantification of the transformation.

Table 1. Van Hiele levels applied to tessellations.

Van Hiele level	Descriptor	Code
Level 0 (Visualisation)	Recognises and names the shapes.	1
Level 1 (Analysis)	Recognises the shapes and describes the transformation informally.	2
Level 2 (Abstraction)	Describes both shapes and transformations unambiguously using technical language.	3

Responses were obtained from 26 Year 5 and 6 students at two Catholic primary schools in Melbourne, Australia. This sample was opportunistic since it depended on the teachers involved in the project. Students coloured in the composite shapes used in each tessellation, and then wrote their descriptions of how the tessellations were made in spaces provided below each design.

Students' responses to each tessellation were coded as shown in Table 1. A code of zero (0) was reserved for an irrelevant or no response.

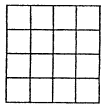
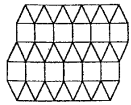
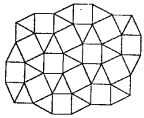
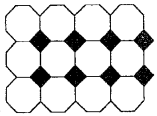
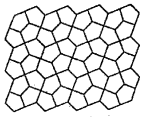
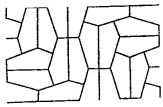
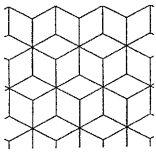
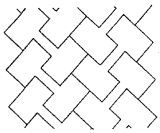
Examine each of the tessellations below. Identify the shapes used in each tessellation. Explain in as much detail as possible how these shapes have been transformed to make each tessellation.	
Tessellation 1 	Tessellation 2 
Tessellation 3 	Tessellation 4 
Tessellation 5 	Tessellation 6 
Tessellation 7 	Tessellation 8 

Figure 1. Tessellation task

RESULTS

The responses of the 26 students to each tessellation are shown in Table 2 as counts and percentages.

The majority of students could describe the tessellations only in a visual manner. This was not unexpected given the age and experience of these primary-aged students. The levels of response shown in Table 2 suggest that as the nature of the shapes of which the tessellation was composed became less familiar the level of response was reduced. Tessellation 1, composed entirely of squares, attracted the highest overall levels of response with nearly half of the students responding beyond the

Visualisation level. In contrast, no students were able to describe Tessellations 6 and 8 at the Abstraction level. These two tessellations were composed of shapes that might be informally described e.g. “fly wings” and “dumbbells”, and they also had some rotational transformation.

Table 2. Responses of students to each question on the tessellation task.

Tessellation	Irrelevant	Visualisation	Analysis	Abstraction
1	4 (15.4%)	10 (38.5%)	5 (19.2%)	7 (26.9%)
2	2 (7.7%)	14 (53.8%)	7 (26.9%)	3 (11.5%)
3	3 (11.5%)	11 (42.3%)	11 (42.3%)	1 (3.8%)
4	4 (15.4%)	17 (65.4%)	3 (11.5%)	2 (7.7%)
5	6 (23.1%)	9 (34.6%)	9 (34.6%)	2 (7.7%)
6	11 (42.3%)	8 (30.8%)	7 (26.9%)	0
7	7 (26.9%)	13 (50.0%)	5 (19.2%)	1 (3.8%)
8	13 (50.0%)	11 (42.3%)	2 (7.7%)	0

The highest levels of response overall were elicited by Tessellation 1, with over one-quarter of the students reaching Abstraction level. An example of the Abstraction level response is:

> The starting shape is a square, they are all the same size, and they have been slid across and then slid down. (id 25)

This response recognises the basic shape, places a condition on this (all the same size) and explicitly describes the movements used in two dimensions to create the tessellation, using the informal technical language (slid) used in the curriculum.

Tessellation 3, an irregular tiling pattern produced some somewhat surprising results. The complexity of the pattern seemed to encourage students to apply some level of analysis, although this was not developed further into Abstraction level. A typical response coded at the Analysis level was:

> I think they started with a square with four triangles on the sides and the[y] overlapped them 12 times to get the shape … (id 11)

The idea of “overlapping” was frequently expressed, suggesting that the students were not seeing this as a tiling pattern so much as a design almost in three dimensions.

The three dimensional aspects of some of these patterns was notable. Tessellation 7 was seen only as a set of cubes by 13 (50%) of the students. Only one student explicitly connected the two- and three-dimensional aspects of this tessellation:

> By using the dimond [sic] it looks like block afeact [effect] (id 5]

This response was coded at the visualisation level. Some students, however, appeared to see three-dimensional shapes in many of the tessellations. One student saw Tessellation 1 as a cube:

> The starting shape is a cube on side view with another on top (id 19)

The way in which this student had coloured the tessellation suggested that this was more than the confusion between the words “square” and “cube”. The response, however, was coded as irrelevant since the student had not identified the shapes involved correctly.

For some students, the movements used to create the tessellation, rather than the shapes involved, were dominant. This response to Tessellation 4 describes only the movement rather than the shapes involved, and this student described many of the tessellations in a similar fashion:

> They have been flipped over and slided. They have drawn triangles and covered all sides. (id 3)

This student has not named the shapes involved correctly, but has a relatively sophisticated view of how the tessellation has been created through flipping and sliding. The response however, despite its complexity, was coded as irrelevant since the shapes were not correctly identified.

A more typical description of Tessellation 4 was:

> I see lots of octagons with diamonds between them (id 4)

Few students recognised Tessellations 5 and 6 as being composed of pentagons. Some students saw the composite hexagonal arrangement in Tessellation 5, but several students referred to the shape in Tessellation 6 as “fly wings”. It is likely that students have had little exposure to a wide range of two-dimensional shapes, other than the regular shapes found in commercial pattern block sets, and thus could not easily identify pentagons.

DISCUSSION

A majority of students in this study described tessellations in terms of the shapes of which they were composed: Visualisation level. This did appear to depend, however, on the nature of the shape. Familiar shapes, such as squares and triangles, were more likely to be named whereas only 50 percent of the students could describe Tessellation 8 coherently in terms of the shapes that comprised the design. Familiar shapes also appeared to support higher level descriptions. In terms of tessellation, however, the familiarity could also be something of a hindrance. Tessellation 4, for example, was described by nearly two-thirds of the respondents in terms of octagons and squares, with no suggestion of transformation.

Few students were able to reach Abstraction level on any tessellation and this result is not surprising given their age and experience. Most of these students were, however, in a position to develop understanding of the properties of shapes with appropriate

teaching intervention since they were able to recognise and name the shapes in most instances, and this provides a starting point for further work on properties.

The complex design of Tessellation 3 appeared to support students to reach Analysis level. Similar comments could be made about Tessellation 2 and Tessellation 6. In order to encourage understanding of tessellation, it would appear that students should experience a wide range of designs, made up of familiar and less familiar shapes and transformations.

Of particular interest, however, were the unusual and unexpected responses. It was surprising that a number of students saw many of these designs in three dimensions. This finding suggests that even in the later years of primary school there are a number of students having difficulty understanding 2D representational conventions. The misconceptions shown by some students could compromise their understanding of the properties of 2D shapes, and subsequent development of geometrical ideas.

Those students who saw only the movement of the shapes, rather than the shapes themselves, were in a position to develop understandings of symmetry beyond that of simple reflection symmetry. They could, however, have difficulty developing understanding of the properties of shapes. In contrast, those students who recognised shapes but made no mention of their transformation could have difficulty developing understanding of symmetry, particularly rotational symmetry which inherently demands some identification of movement.

Using the van Hiele levels as a basis appeared to provide a useful means of identifying primary students' understanding of tessellation, that included both recognition of the shapes involved and the transformations used to make the design. It did not, however, adequately identify the nature of the misconceptions shown by some students. This is one limitation of using van Hiele levels, since the types of misconceptions shown potentially could affect students' development of geometrical understanding.

CONCLUSION

It appeared that the majority of students in this small exploratory study could recognise and describe a range of regular shapes when these were part of a tiling pattern. Their recognition of geometrical transformations, however, was more limited, particularly when the shapes involved were less familiar. There were also some unexpected and unusual responses, which might have an impact on students' future understanding. This has implications for the use of tessellation as a means of developing geometrical insight. Teachers should not assume that all students visualise tiling patterns in the same way.

Although a majority of students responded in expected ways, some appeared to view tessellation only in terms of movement, and others saw the tilings in three- rather than two-dimensions. These unusual responses have implications for teaching. If tessellation is used as a means of developing students' geometrical knowledge of the properties of shapes, and of symmetry and transformation, then opportunities are

needed for all students to learn about both aspects of tessellation: the nature of the shapes involved and also the kind of transformation applied.

The mathematics of tessellation is not generally well developed in the school curriculum. This initial study suggests that young students do not necessarily respond to tiling patterns in expected ways. Further research into older students' comprehension would be useful to explicate a developmental sequence of understanding of tessellation. In addition, further research into students' apparent misconceptions would provide useful information relating to the development of ideas about symmetry and transformation.

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FACILITATING PEER INTERACTIONS IN LEARNING MATHEMATICS: TEACHERS' PRACTICAL KNOWLEDGE

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This paper reports on teachers' practical knowledge [PK] about peer interactions [PI] in learning mathematics. The focus is on high school teachers who consistently engaged students PI in their teaching. Data consisted of interviews and classroom observations. Findings indicate that these teachers have PK of students' roles in PI and learning activities and teacher's behaviors to support PI that creates a meaningful classroom culture to facilitate PI in learning mathematics. Their classroom experiences and their conceptions of mathematics and learning played an important role in the PK. Their PK offers insights into pedagogical strategies that can be effective in facilitating PI. PK is a basis for teachers' sense making and can play an import role in teacher education.

There are reform recommendations in mathematics education that assign a significant role to peer interactions in teaching and learning mathematics. For example, the National Council for Teachers of Mathematics [NCTM] standards advocate:

> Whether working in small or large groups, they [students] should be the audience for one another's comments – that is, they should speak to one another, aiming to convince or to question peers [NCTM 1991, p.45].

However, whether or how peer interactions get implemented in the classroom will likely depend on the teacher. Thus, the study in this paper focused on understanding peer interactions through the teacher. In particular, it investigated teachers' practical knowledge of peer interactions in learning mathematics in terms of how the teachers made sense of peer interactions and when and how they incorporated it in their teaching.

RELATED LITERATURE AND THEORETICAL PERSPECTIVE

Studies on mathematics teachers have examined their content knowledge, beliefs, conceptions, classroom practices, learning, professional development and change (e.g., Chapman, 1997; Fennema & Nelson, 1997; Lampert & Ball, 1998; Leder et al., 2003; Schifter, 1998; Thompson, 1992). These studies have provided us with insights on, for example, the relationship between beliefs/conceptions and teaching, deficiencies in teachers' content knowledge, and the challenges of teacher education and teacher change. In particular, research on the mathematics teacher suggests that an understanding of teachers' thinking and actions are important to improve the teaching of mathematics. Boaler (2003) argued that researchers need to study classroom practices in order to understand the relationship between teaching and learning. The teacher is the determining factor of how the mathematics curriculum is interpreted and taught. Thus, it is important to learn from teachers what they do and how they make sense of what they do in the classroom. In this paper, the focus is on the teacher's perspective of peer interactions in the teaching and learning of mathematics.

The research literature provides a lot of theory on group or cooperative learning (e.g.,

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Davidson, 1990; Slavin, 1995), but there is little attention to the teacher's perspective of it. Davidson provides several examples and "practical strategies" for using cooperative groups in mathematics teaching and learning. It is not clear what conceptions underlie teachers' thinking that will help or hinder their integration of such approaches in their teaching. Investigating their practical knowledge is one way of making this explicit.

Theoretically, the study is framed in a social or interactive perspective of learning and a practical knowledge perspective of teacher thinking, each of which is briefly described here. An interactive perspective on teaching and learning has been discussed by several people including Bauersfeld, 1979; Dewey, 1916; Lave & Wenger, 1991; and Vygotsky, 1978. Lave and Wenger conceived of learning in terms of participation. Dewey emphasized learning through active personal experience and learning as a social process. In his view, purposeful activity in social settings is the key to genuine learning. Vygotsky claimed that individual development and learning are influenced by communication with others in social settings. In his view, interacting with peers in cooperative social settings gives the learner ample opportunity to observe, imitate, and subsequently develop higher mental functions. Specific to mathematics, Bauersfeld (1979) explained:

> Teaching and learning mathematics is realized through *human interaction*. It is a kind of mutual influencing, an interdependence of the actions of both teacher and student on many levels. … The student's reconstruction of meaning is a construction via social negotiation about what is meant and about which performance of meaning gets the teacher's (or peer's) sanction. [p.25]

This theoretical perspective, then, promotes the position that learning takes place in a social setting and emphasizes human interactions as a key factor to facilitate learning. One way in which this perspective has been conceptualized in relation to the classroom is in terms of cooperative groups. Such groups have been promoted as having three key goals: to distribute classroom talk more widely, encouraging students to talk, to share their ideas and to become more actively engaged; to specify the social processes to help students to work cooperatively; and a way to develop their social and collaborative skills. However, like any tool, how these goals are interpreted and applied by teachers may or may not fit the interactive perspective beyond an instrumental level. In this study, peer interactions are considered to be classroom situations where students talk with students directly to learn mathematics. This includes small-group or whole-class situations but excludes situations where the teacher mediates a discussion, e.g., the teacher asks a student to react to another student's response or a student initiates the response but directs it to the teacher who acts as a bridge between the students.

The second aspect of the theoretical perspective framing this study is the construct of practical knowledge [PK]. PK has been used in research of the teacher to describe knowledge that guides teachers' actions in practice (Johnston, 1992). Whereas scientific or formal knowledge is abstract and propositional, PK is experiential, procedural, situational, particularistic, and implicit (Carter, 1990, Fenstermacher, 1994). It corresponds with positions teachers take. It refers to teachers' knowledge of classroom

situations and the practical dilemmas they face in carrying out purposeful action in these settings (Carter, 1990). Experience is an important source of PK.

PK in this study is based on Sternberg & Caruso's (1985) theory of it. In their view, PK is procedural information that is useful to one's everyday life [which includes teaching, in the case of the teacher]. It is used in three main forms of interaction with the everyday world – adaptation, shaping and selection. For example, a teacher would use PK to adapt to situations in the classroom, or to shape situations in the classroom or to make selections when choices are available. PK is stored as or conveyed by statements that embody "condition-actions sequences"(p. 135), i.e., if I do A (the condition), B (the action) will happen. In the case of teaching, such statements describe the procedure or condition that will bring about a certain action or performance state in students. For example: I know that students will not improve their achievement in mathematics if I use groups. PK can be probabilistic in nature. For example: If I use groups, it probably wouldn't make a difference. Sternberg & Caruso explain, "Knowledge becomes practical only by virtue of its relation to the knower and the knower's environment (p. 136)." This implies that a teacher's PK is relevant to his or her personal context or classroom context. The goal of this study, then, is to identify conditions and actions for peer interactions that are common to a sample of teachers to understand how this process makes sense to them.

RESEARCH PROCESS

The data for this study is based on a larger study on teacher thinking in teaching word problems framed in a phenomenological research perspective (Creswell, 1998) that focuses on participants' meaning, what they value, and how they make sense of their experiences. The participants were 22 elementary, junior high and senior high school mathematics teachers from local schools. The main criterion for selecting the teachers was willingness to participate. However, most of the ten high school teachers were considered to be exemplary teachers in their school systems. Some had won teaching awards. All of the participants were articulate and open about their thinking and experiences with word problems and mathematics.

The main sources of data for the study were open-ended interviews and classroom observations. Interview questions were framed in a phenomenological context to allow the teachers to share *their* way of thinking and to describe their behaviors as lived experiences (i.e., stories of actual events). The interviews focused on their thinking/ experiences with word problems in three contexts: (i) past experiences, as both students and teachers, focusing on teacher and student presage characteristics, task features, classroom processes and contextual conditions, (ii) current practice with particular emphasis on classroom processes, planning and intentions, and (iii) future practice, i.e., expectations. Questions were often in the form of open situations, e.g., telling stories of memorable, liked and disliked classes involving word problems that they taught, giving a presentation on word problems at a teacher conference, and having a conversation with a preservice teacher about word problems. Because of the open-endedness of the

questions, their responses extended beyond word problems to their teaching of mathematics in general. Classroom observations over a 2-week period for each teacher, focused on the teachers' actual instructional behaviors during lessons involving/related to word problems. Special attention was given to what the teachers and students did during instruction and how their actions interacted. Post-observation interviews with each teacher focused on clarifying her/his thinking in relation to her/his actions.

The analysis began with open-ended coding (Strauss & Corbin, 1998) of the audio-taped transcripts of interviews. The coding was done by the researcher and two research assistants working independently to identify attributes of the teachers' thinking and actions that were characteristic of their perspective of teaching word problems. The focus was on significant statements and actions that reflected judgments, intentions, expectations, and values of the teachers regarding their teaching that occurred on several occasions in different contexts. The coding was followed by a review of the field notes and audio-taped transcripts of classroom observations to triangulate the findings from the interviews and add and clarify situations. This was followed by comparison of the findings by the three coders and revisions made where needed. The coded information for each participant was then sorted into themes that conveyed the significant features of his/her thinking and teaching. Peer interaction was one of the themes that emerged. In order to elaborate on this theme, the researcher and two assistants returned to the data to obtain details of all situations that conveyed it and to identify the conditions and actions that constituted the teachers' PK about peer interactions. Verification procedures, then, included triangulation, using data from a variety of sources, cross checks by research team, and elimination of initial assumptions/themes based on disconfirming evidence.

PRACTICAL KNOWLEDGE OF PEER INTERACTIONS

The findings reported here focus only on those teachers who consistently provided opportunities for students to engage in peer interactions. These were the eight high school teachers who were identified as exemplary in teaching mathematics. There was more depth and scope to their PK than that of the other teachers. Their PK was influenced by their experience as a teacher, for e.g., they recognized that teachers and students vary in their explanations of the same content. For example,

> I don't see things the way kids see things, and I don't solve problems the way kids solve problems, … If you're explaining something, they can sit and look at the board and you can tell they don't get it … so you ask somebody else in class and they might say exactly the same thing I've said …and then the kid will go, yes that's right, I understand, and you're there, like, but I just said that. … Somehow they know how to relate it to each other, and many times they can express things in different ways that I haven't thought of.

Their PK was also influenced by their own experience with peer interactions, for e.g.,

> I need to talk and often when I talk some of my best ideas come out. Writing about it doesn't always do it for me, so I think the peer interaction can do that.

Conditions and actions for four themes characterized the teachers' PK of peer interactions: conceptions [conditions] that support a social perspective; students' behaviors/outcomes [actions] for/from peer interactions; learning activities [conditions]

to support peer interactions; and teacher's behaviors [actions] to support peer interactions. Each is discussed with examples of representative quotes from the teachers.

Conceptions That Support a Social Perspective: The teachers' PK included a view of mathematics and learning that supported a social perspective of mathematics education and embodied the need for peer interactions in their teaching. For example:

- Math is primarily a problem solving activity and problem solving is a social activity.
- Mathematics to me is a shared experience.
- I believe math is a language and to be able to articulate it is a very important part of the learning process.
- Learning comes through talk, comes through discussion.
- Mathematics learning occurs when the learner understands and can explain the concept that has been presented in their own words … and knows it sufficiently to teach to someone else, talk about it to someone else.

Students' Behaviors/Outcomes for/from Peer Interactions: The teachers' PK indicated that through peer interactions, students learn mathematics from and with each other as they engage in/achieve the following seven behaviors/outcomes.

Compare experiences: This allows students to learn about learning. For example,

Getting information from your peers helps you understand that there are people who are experiencing the same difficulties as you or have a perspective that you can share.

Share ideas: This allows students to collaborate and expand their thinking. For e.g.:

- They can bring something to this situation that you may not have thought of.
- Sometimes a student just doesn't understand, and then another student will go well what about such and such, and together they can formulate a conclusion. Whereas, individually, they would have been stuck.
- Just by people sharing each other's views, they help refine each other's views.

Articulate mathematics: This involves students orally expressing mathematics in words or describing and explaining it (e.g., concepts) in a meaningful way. For e.g.:

- They have to explain in plain English what something means.
- They talk math language … they use the language and each other understands more than they do with the teacher.
- By having the students explain, they put it in words or they put it in a context that is more meaningful than I have done.

Pose questions: This involves students "asking each other questions" and allowing

the rest of their group to question them and to debate with them whether or not what they've told the group is valid or whether or not that's taking them down the garden path.

Be motivated and gain confidence: For example,

- They motivate each other and help build confidence.
- They lend support to each other and … motivate each other to get their work done.

Gain autonomy: This involves students depending less on the teacher's thinking. For e.g

They won't always look to the teacher for solutions, they'll look to each other, … but also they get to interact more with each other, and can use each other more to enhance their own learning.

Test understanding: For example,

- It's only through peer interaction that you get to test out your thoughts, your ideas, that

> you get to formulate your own perception of the mathematics that confirms for you that you understand it.
> - It's by saying, "Well now, how did you get that?" My answer doesn't look like that. It's with peers beside you that they start to discuss and compare their work, compare the answers, compare the steps and they promote each other's learning and understanding.

Learning Activities That Support Peer Interactions: The teachers' PK indicated that by engaging students in the following five learning activities/experiences, they will have opportunities to engage in peer interaction. The focus, as one teacher noted, is to "encourage students to investigate and solve mathematical problems with each other."

Inquiry of the problem-solving process: The teachers allowed students to work in groups to learn about problem solving. For example, some teachers presented students with a problem to solve on their own. More importantly, they were required to analyze their process by considering, for e.g., what they did, why used that method, why it worked, how they made sense of the problem and the solution. This is followed by group sharing and whole-class discussions. The process is repeated for a few problems to establish a model for solving problems. One teacher had students work in groups to solve the problem, but one student in each group assumed the role of observer and "looks at the process that is going on to solve the problem". Each student got a turn at being observer. A whole-class sharing and comparison of findings followed each round of observations. This process eventually led to the development of a problem-solving model.

Inquiry of a new concept: The teachers used peer interactions in a variety of ways in introducing a new concept. The most common approach was to allow students to explore a situation in small groups before the teacher provided any explanations or led a discussion on it. The situation could involve "just looking for patterns and relationships" to make sense of a concept or trying to solve a problem to understand a procedure. For example, in introducing systems of equations, one teacher presented a scenario involving weights of large and small cats by drawing a picture of it on the board and asking students to work in their groups to find the weights of the cats. She gave no other direction. After students shared and compared their approaches, she connected them to the formal approaches. Another teacher assigned each group a different method of solving systems of equations and required that they analyzed solved examples to understand the method. Each group then taught the method they explored to the other groups, trying to convince each other that their method was the best approach.

Whole-class presentation: The teachers encouraged students to interact during whole-class presentation. The common approach was that, students' did not only present and justify their solutions, but, for example:

> They have to encourage others, okay now what did you use, and how did it work, how did you do it and which method do you think you like better, and why.

One teacher, as an introduction on systems of equations, had students collect pictures of real-life situations of graphs that intersect and take turns to lead a discussion about

> What the graph represents, the significance of the graph, what it means when graphs intersect, what the intersection shows, why the intersection is important, why anyone

would want to find the intersection in the first place.

Practicing problem solving: Students practiced problems individually or collaboratively, but always had the opportunity to discuss their work with peers.

Investigations/projects: Students planned and conducted group projects that included the use of technology, use of art and outdoor activities, e.g., finding heights of tall objects.

Teacher's Behaviors to Support Peer Interactions: The teachers' PK indicated that the following four behaviors of the teacher will facilitate or support peer interactions that promote learning.

Listens and observes: This provides the teacher with feedback to determine when and how to intervene. For example:

> I circulate and I listen to the kinds of conversations that go on in the groups and how are they processing the information, how are they developing the strategies that they are going to use and try and get some clues on what they understand.

Questions and prompts: This involves the teacher using questions to extend concept development or check for understanding or prompts when students are stuck. For e.g.:

> - I ask them questions if they're stuck but that's about it. … I will try to come up with a question that will allow them to continue but I will not give them the answer at any time.
> - My role is to ask questions, not to give answers. … It is by how you ask the questions that they gain access to the answer that they wanted you to just tell them.

Supports students' thinking: For example,

> - I make sure that I tell them that I don't care how they solve it, but they have to say exactly how they did this.
> - Make them aware that they have tools, mathematical tools, then give them the freedom to use whatever they want to solve these problems.
> - I always tell students I am not the only expert in the classroom and I learn from you just as well as you learn from me.

Models questioning: This involves the teacher using a questioning approach during whole-class instruction that students then mirror. For example: I notice they ask

> themselves the same questions when they're group working on problems.

Promotes good peer relations: As one teacher explained, "If you can facilitate good peer relations then you can have some really healthy dialogue." Some ways in which this occurred were through shared questions, seating plan, voluntary grouping, and peer observations:

> - You can only ask me a question when all of you [in a group] share the same question.
> - I sit them beside each other in groups. There's no time that they're ever in isolated rows or isolated desks. …I don't force the groups because I think that just creates a social conflict when you're trying to deal with the math conflict.

A unique approach used by one teacher was having each student take turns observing his/her group behaviors as the group solved genuine mathematics problems.

> They are looking at how the actual group interaction occurs so they do become a good cooperative learning group and they support … and work with each other.

CONCLUSION

The findings indicate that PK is a basis for the teachers' sense making in using peer interactions [PI] in their teaching. These teachers have PK of PI that facilitates a

meaningful classroom culture to support the learning of mathematics. Their experience provides evidence that this instructional approach is a feasible and meaningful way of teaching high school mathematics. The findings also suggest that the teachers' classroom experiences and conceptions of mathematics and learning play an important role in characterizing their PK. This has implications for teacher education. For e.g., positive personal experiences with PI may be necessary for teachers to develop meaningful PK. Exposure to PK of teachers like these participants could open the door to gaining such experience. Their PK offers examples of the way things are or could be. Also, since all teachers, including preservice teachers, have some PK used as a basis of sense making of PI, providing them with theory on PI that conflicts with their PK could be problematic for them if their PK is not explicitly dealt with. The findings of this study offer a structure against which other teachers could examine their PK, either through reaction against or resonance with what is offered, to understand it. Finally, this conception of PK can be used to study other aspects of teaching mathematics.

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TOWARDS A UNIFIED MODEL ON TEACHERS' CONCERNS AND EFFICACY BELIEFS RELATED TO A MATHEMATICS REFORM

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Teachers' concerns and efficacy beliefs (EB) are important for the success of any reform. Previous research has shown that teachers' concerns develop in three levels: self, task and impact, respectively. Thus, this study examines the concerns and EB of primary teachers with respect to a reform concerning the use of Schema Theory in teaching problem solving (PS). A proposed model connecting teachers' EB and concerns is also tested. Analysis of data suggests that teachers' concerns were situated in the first level; teachers seemed to feel more efficacious in teaching PS without using the reform. Teachers' concerns were affected by their EB, which in turn, were affected by first-level concerns. Concerns of succeeding levels were also influenced by concerns of preceding levels. Implications of findings for the development of the reform policy and for further research are drawn.

INTRODUCTION

Following international trends, a reformed primary mathematics curriculum was introduced in Cyprus in 1998. One of the characteristics of the reform was the use of a model for solving problems, mainly based on Mashall's Schema Theory (1995). The model was in operation for the last five years, during which the teachers involved expressed contradictory evaluations about the practical usefulness of the model to enhance students' problem solving (PS) ability.

Research findings during the last two decades underlined the importance of examining teachers' reaction towards a reform; any change is associated with new demands on the part of the teacher and naturally the success of any reform effort depends highly on the teachers' role (Amit & Fried, 2002; Sztajn, 2003). Two research domains that have been studied excessively during the last three decades, i.e., teachers' concerns and teachers' efficacy beliefs (EB) are nowadays revisited and connected to teachers' attitudes towards the implementation of a reform, as well to each other. Research has shown that these constructs influence teachers' attitude towards a reform and their attempts to implement it (Tschannen-Moran, Woolfolk-Hoy & Hoy, 1998; Piggie & Marso, 1997).

Fuller introduced the concept of teachers' concerns in the late 1960s (van den Berg & Ros, 1999) and put forward a classification of teachers' concerns consisting of three developmental levels, namely self, task and impact concerns. Self-concerns mainly relate to the teachers' anxiety about their ability to take over the new demands in the school environment, task concerns refer to the daily duties of a teaching job, especially in relation to a number of limitations such as time constraints, teaching a large number of students or the lack of resources. Finally, impact concerns deal with

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teachers' apprehension concerning students' outcomes. Incorporating Fuller's conceptualization, the Concerns Based Adoption Model (CBAM) (McKinney, Sexton & Meyerson, 1999) identifies seven stages of concerns: awareness, informational, personal, management, consequences, collaboration and refocusing. The first three stages constitute self-concerns, the fourth relates to task concerns and the three latter represent impact concerns. According to the model, initially teachers have little knowledge of the innovation (awareness); later on, they are concerned about their ability to respond to the requirements of the reform (personal) and they show their willingness to learn more about it (informational). Self-concerns gradually decrease and teachers focus on managing the reform (management). Finally, teachers overcome tasks concerns and focus upon the effects of the reform on students' learning (consequences) and seek for cooperation with their colleagues (collaboration); they also make suggestions for improvements regarding the reform (refocusing). Though there was evidence about the developmental nature of teachers' concerns (e.g., van den Berg & Ros, 1999; Piggie and Marso, 1997), to the best of our knowledge, no systematic attempt has been undertaken to test the assumptions of the CBAM.

The construct of efficacy beliefs (EB) was initiated in the 1970s and refers to one's ability to plan and execute actions to achieve a goal (Bandura, 1997). EB were found to exert great influence in adopting and implementing an innovation; teachers possessing high EB harbor more positive attitudes towards the innovation, are more likely to implement it and regard it as important and compatible with their traditional way of working (Tschannen-Moran et al., 1998). Moreover, these teachers are more willing to experiment with new teaching approaches and materials, and are less anxious about the reform and the possible limitations or complications deriving from it (Bandura, 1997).

Recently, research has employed both these concepts to study the implementation of a reform; it has been shown that there is an interaction between teachers' EB and their concerns about the reform. Specifically, teachers with low EB have been found to display intense self and task concerns compared to their high efficacious colleagues (Ghaith & Shaaban, 1999). Moreover, the more efficacious teachers are feeling regarding the innovation, the more intense are their impact concerns (McKinney et al., 1999). It has been also found that teachers' concerns are largely affected by their EB (Christou, Philippou, Pitta-Pantazi & Menon-Eliophotou, 2002). However, researchers have moved only in one direction focusing on the extent to which EB affect teachers' concerns. It can be claimed that teachers' concerns, especially those related to awareness about the reform, may influence their level of EB. This assumption is based on the fact that research reveals that teachers' knowledge in a specific domain influence their efficacy in teaching subjects related to this domain (Wenner, 2001).

In the light of the above, the purpose of this study was twofold. It aimed to examine teachers' concerns and EB in teaching PS by using the reform model and to develop a

model connecting teachers' EB and concerns. Specifically, three hypothesis were tested: (a) teachers' concerns can form a hierarchical model (awareness, informational, personal, management, consequences, collaboration and refocusing), with preceding stages affecting teachers' concerns in subsequent stages, (b) EB affect teachers' second and third level concerns (task and impact concerns), while they are affected by their first level concerns, and (c) teachers' EB about employing approaches used prior to the reform, affect their concerns about the reform.

METHODS

Stratified sampling was used to select 27 (rural and urban) primary schools in Cyprus. Since the new PS model is introduced in the fourth grade, and employed henceforth, the teachers of the aforementioned schools who were teaching at the three upper grades completed a questionnaire of 52 items on a nine point Likert scale that reflected their concerns and EB as regards the specific reform. Specifically, was used. Thirty-seven items derived from the Stages of Concerns included in the CBAM, translated in Greek and reworded to reflect the characteristics of the specific reform; the remaining 15 statements referred to teachers' EB. Responding to the need to increase the specificity of efficacy items (Nielsen & Moore, 2003), statements were developed to measure teachers' EB in teaching PS either by using the new model or by employing traditional strategies used prior to the introduction of the reform. The response rate (90.4%) was very high, since 151 out of the 167 teachers completed the questionnaires.

The data were initially analyzed through exploratory factor analysis, which identified non-directly observable factors based on teachers' responses. Structural equation modeling was next employed to test the hypotheses of the study. Maximum likelihood method was used to estimate the model parameters, since this method does not require data from extremely large samples (Kline, 1998). More than one fit index was used to evaluate the extent to which the data fit the models tested. Specifically, the scaled chi-square, Bentler's (1990) Comparative Fit Index (CFI), and the Root Mean Square Error of Approximation (RMSEA) (Brown & Mels, 1990) were examined.

FINDINGS

Exploratory factor analysis resulted in a seven-factor solution explaining 62.45% of the total variance. The seven factors were related to teachers' EB and concerns and were identical to those mentioned in the specification table of the questionnaire. This finding provides support to the construct validity of the questionnaire used to collect data on teachers' EB and concerns about the reform (Cronbach, 1990). Thus, factor scores for each dimension were estimated, by calculating the average of the items that comprised each factor. The mean scores, the relevant values of standard deviations and Cronbach's alpha values for each factor are presented in Table 1.

No	Factors	$\bar{x}$*	SD	A
	Factors related to teachers' EB			
1.	EB in teaching PS without using the model.	7.74	0.94	.77
2.	EB in teaching PS by using the model.	6.49	1.46	.92
	Factors related to teachers' concerns:			
1.	Need for Information.	6.73	1.70	.81
2.	Awareness	5.51	1.81	.81
3.	Refocusing-negative criticism towards the model.	5.37	1.93	.87
4.	Consequences on students	5.11	1.86	.89
5.	Management	4.70	1.47	.69

* In a 9 point-scale (1=negative and 9=positive)

Table 1: Means, Standard Deviations, and Cronbach's alpha coefficients of the seven factors identified by exploratory factor analysis.

Table 1 shows that the values of Crobach's alpha of six factors are relatively high (i.e. ranging from .77 to .92). This implies that the measurement errors of these six scales are relatively low and thereby the collected data on the six factors can be considered reliable. On the other hand, the reliability of the "management" factor is moderate satisfactory (.69). Nevertheless, this can be attributed to the procedure used to estimate Cronbach's Alpha, which is highly dependent on the number of items of each scale (Norusis, 1993). Thus, the low value of the Cronbach Alpha of this scale is partly due to the fact that only four questionnaire items were used to measure this factor. Table 1 also suggests that teachers' level of EB (either by using the model or without it) was quite high. However, the paired t-test revealed that their EB in teaching PS without using the reform model were significantly higher than their corresponding beliefs in using the reform ($t=10.40$, $df=136$, $p<.001$). Table 1 also shows that teachers experienced intense concerns related to the level of the information they have received about the reform. Though teachers adopted a rather critical approach to the model ($\bar{x}=5.37$), in general this opinion was not strong enough, since its standard deviation was relatively high (SD=1.93). Similarly, concerns about the consequences of the model on pupils and the management of the reform appeared moderate. In general, it can be argued that five years after the introduction of the reform, teachers' concerns were mainly situated in the first level of concerns (awareness and information).

Kendall's test was subsequently employed in order to rank the different types of teachers' concerns, based on the level of their intensity. Specifically, the Kendall Coefficient of Concordance was calculated and revealed a significant level of agreement among teachers about the intensity of different types of concerns ($w=.21$, $p<.001$). Informational concerns were placed at the "intense" end of the scale (mean rank, mr =4.16), along with the concerns reflecting teachers' awareness of the reform (mr=3.15). The refocusing concerns were somewhere in the middle (mr=2.86), while at the least "intensive" end one could identify teachers' concerns regarding the consequences on pupils (mr=2.45) and managing the reform (mr=2.38).

The factor scores of the seven factors identified above were also employed to search for the relationships between teachers' concerns and EB, using the EQS (Bentler, 1995). As reflected by the iterative summary, the goodness of fit statistics showed that the data did not fit the model very well (x^2=14.12, df=6, p<.03; CFI=.978, and RMSEA=.170). Subsequent model tests revealed that the model fit indices could be improved by adding another path joining management concerns and refocusing. The model that emerged after this modification had a very good fit to the data (x^2=8.25, df=7, p>.23; CFI=.997 and RMSEA=.036). Figure 1 shows the model that emerged, as well as the path coefficients among the seven factors. The following observations arise from Figure 1. First, teachers that held high EB to teach PS without using the model tended to support that they had received more information regarding the reform. On the other hand, the more aware teachers were about the reform, the higher their EB were to teach by using the model. Teachers' EB to teach PS by using the reform were also explained by their efficacy to teach PS without using the model. Second, the more aware teachers were about the reform, the lower their need was to get more information as regards the underlying theory, philosophy and aims of the reform.

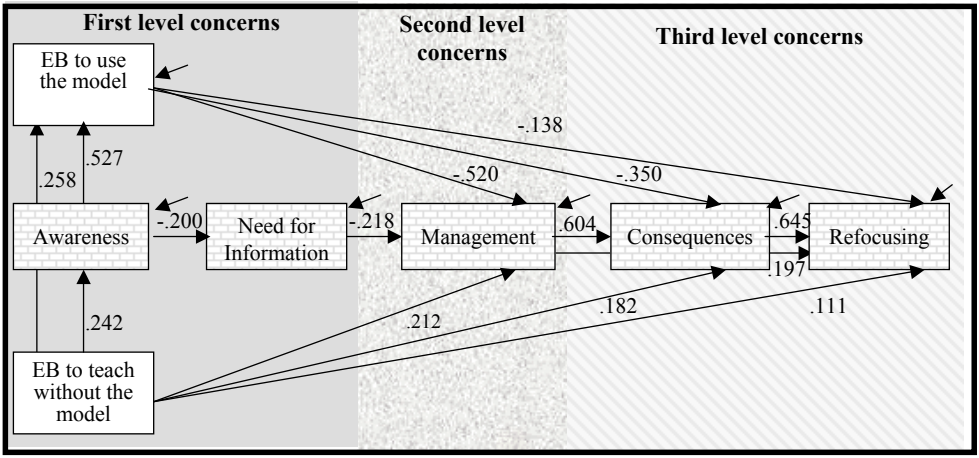

Figure 1: Path model of the seven factors linked to teachers' efficacy beliefs (EB) and concerns regarding the implementation of the problem solving reform.

Third, teachers' second level concerns (management concerns) were explained both by their concerns of the preceding stage as well as by their EB. Namely, teachers who supported that they had received a satisfactory level of information regarding the reform, and who reported a high level of efficacy to teach PS by using the model were less concerned about issues related to managing the reform. On the contrary, teachers who conceived themselves as highly efficacious in teaching PS without using the model were highly concerned regarding the management of the reform.

Fourth, teachers' third level concerns were explained by their second level management concerns as well as by their EB. Specifically, teachers that harbored high EB in teaching by using the reform, low EB in teaching PS by employing approaches used prior to the introduction of the reform and who reported low concerns regarding the management of the model were less concerned about the consequences of the reform on pupils. It should also be noted that management concerns explained more variance of teachers' concerns about the consequences of the reform on pupils than their EB to teach PS either by using the model or without it. The same pattern was also identified in the case of refocusing concerns, though, in a revered order. Namely, the more efficacious teachers tended to be in teaching PS without the model and the least efficacious they felt in using the model, the stronger they supported the need to abolish the model and resort to previously used PS teaching approaches. Moreover, even though management and refocusing concerns did not appear subsequently, the higher teachers' concerns were regarding the management of the reform, the more they were disfavor of the reform.

In general, the model of Figure 1 verified the three examined hypotheses: The factors related to concerns were found to form a hierarchical model, with every preceding stage explaining a proportion of the variance of the subsequent stage. On the other hand, EB affected teachers' second and third level concerns. Finally, teachers' level of awareness was found to influence their EB to teach PS by using the reform, whereas these concerns were affected by teachers' EB to teach PS without using the model.

DISCUSSION

The findings of the study reveal that even five years after the implementation of the reform Cypriot teachers mainly exhibit self-concerns. Namely, teachers were more concerned about the level of their awareness about the reform; thereby they expressed intense concerns about the need to receive more information about the reform. This finding is in line with previous research suggesting that self-concerns are not quickly solved and that it may take three to five years before teachers move from this level of concerns to the next one (van den Berg & Ros, 1999). Teachers were also found to harbor relatively positive EB about PS, either by using the model or by resorting to teaching strategies used prior to the introduction of the model. This finding seems reasonable, taking into account the emphasis given to PS in Cyprus (Charalambous, Kyriakides & Philippou, 2003). However, it should be recognized that teachers harbored a higher level of EB in teaching PS without using the model, justifying previous findings (Fullan, 1991; Ghaith & Shaaban, 1999) according to which teachers feel more efficacious in using tested and tried methods than employing innovative approaches in their teaching.

The present study also provided support to the assumption that the various stages of teachers' concerns can form a hierarchical model, since it was found that teachers' concerns in succeeding stages were influenced by their concerns in preceding stages. However, the path connecting management and refocusing concerns identified in the

study supports that teachers might simultaneously experience concerns of different levels. This is in line with recent findings (Burn, Hagger, Mutton & Everton, 2003) and raises doubts about the existence of a developmental scale able to discriminate teachers in distinct stages of concerns. Further research could elaborate more on this assumption, testing the possibility of placing teachers into different levels representing a mixture rather than discrete concerns. Specifically, item response theory models could be used to examine the separability of the person estimates scale (Charalambous et al., 2003).

Moreover, the study illustrated the important role of EB in the implementation of a reform. Teachers holding higher EB in using the reform model were found to experience less worries about issues related to the management of the reform and the influence of the reform on students' achievement; they were also less critical about the reform. It should be noted, though, that teachers' efficacy was affected by their level of awareness about the reform, indicating that a first step in developing teachers' efficacy is to provide them ample information about the philosophy and aims of the reform. In the present study a new element was also added to the efficacy and concerns model, i.e., teachers' efficacy in using strategies employed before the introduction of the reform. This type of teachers' efficacy exerted the same influence on their concerns, but in a reversed way, suggesting that teachers who feel efficacious to teach by using certain teaching strategies criticize a reform more and foresee more problems as a result of its implementation. In sum, the findings of this study are in line with previous findings showing that EB are important factors to be considered in efforts to initiate and sustain educational change (Mc Kinney et al., 1999).

In the light of the above, policy makers could try to improve teachers' awareness about the reform, by providing them ample information related to it. By doing so, they make the first step to advance teachers' efficacy, which subsequently may help teachers envision fewer problems when requested to implement a reform and make them less critical to it. Reacting positively towards a reform is indeed important, taking into account teachers' resistance to mathematics reforms (Amit & Fried, 2002). It should be finally indicated that future research needs to cross-validate the model that emerged from the present study, both by looking at different forms of changes in mathematics and by collecting data in different educational settings.

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WHAT IS UNUSUAL? THE CASE OF A MEDIA GRAPH

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Three hundred and twenty-four middle school students considered a group of three graphs in a newspaper article about boating deaths. The graphs contained discrepancies and the students were asked to "comment on unusual features." This form of questioning produced a distribution of responses surprising to the authors and perhaps challenging to current goals for statistical literacy. Of these students, 201 answered the same question two years later and although overall performance improved to some extent there were still very few high level responses. The outcomes point to specific suggestions that can be made for middle school classrooms in line with the goals of statistical literacy.

INTRODUCTION

Quantitative literacy and critical numeracy have emerged as avenues for considering mathematics in a reform curriculum aimed at catering for all students (Steen, 2001); in the same way statistical literacy is taking the chance and data curriculum to a wider audience. Adults need to interpret the information with which they are inundated daily; but what are the criteria for effective decision-making? International adult literacy surveys (e.g., Dossey, 1997) have considered document literacy and quantitative literacy alongside prose literacy as significant tools required by adults in western society. The tasks employed in these surveys have a strong reliance on statistical ideas, particularly graph interpretation.

Gal (2002, pp. 2-3) suggested that statistical literacy considers people's ability to interpret and critically evaluate statistical information, and their ability to communicate their understanding, concerns, and reactions. Watson (1997) proposed a three-tiered hierarchy for statistical literacy, incorporating (i) an understanding of basic statistical terminology and tools, (ii) an understanding of these terms and tools within societal contexts, and (iii) the ability to question claims made without appropriate justification. These steps are similar to the code-breaking, text-meaning and usage, and critical thinking components associated with models of critical literacy (e.g., Freebody & Luke, 2003).

The current study arose from a larger project that was focused on school students' appreciation of variation as the foundation of the chance and data curriculum (see, e.g., Watson & Kelly, 2002, 2003). Although the aim of the larger study was to describe the development of the understanding of statistical variation, tasks were developed in contexts that employed the specific topics in the curriculum, such as chance events, averages, and graphs, and thus considered aspects of statistical literacy as well. The contexts varied, including simple settings such as rolling a die, familiar settings such as a school survey, and unfamiliar social settings such as would be

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found outside of school. It is for this last setting and the topic of graphs that the task discussed here was devised.

The graphs upon which the task was based are shown in Figure 1 (Haley, 2000) and were chosen for their potential for measuring aspects of statistical literacy in a context involving authentic variation. The straightforward bar graph style was considered accessible to all students at the middle school level. Having three graphs instead of one allowed features to be compared and contrasted. The context for the graphs—boating deaths in the state where the students lived—was considered comprehensible, as well as a social issue worth considering in terms of the goals of statistical literacy. There were two anomalies in the graphs that would allow students to question and display critical statistical literacy skills. There was also variation in the graphs, which was an underlying feature of the larger study.

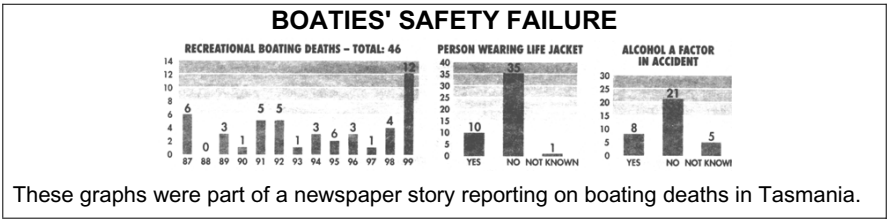

Figure 1: Set of three graphs used in the task (Haley, 2000)

Of interest was what students would attend to when examining the graphs. What aspects of the graphs would they find "unusual"? Would they be influenced by the authentic nature of a newspaper extract and be unwilling to question it? Specifically, this study examines the categories of response that characterise middle school students' descriptions of unusual features of bar graphs from the media (containing technical discrepancies). It also considers whether the responses change over a two-year period.

METHODOLOGY

Sample. The sample consisted of 156 students in Grade 7 (age 12-13) and 168 students in Grade 9 (age 14-15) at four government high schools in the Australian state of Tasmania. Of these students, 137 in Grade 7 and 64 in Grade 9 responded again two years later. Fewer students were surveyed in Grade 11 because many leave or change schools at the end of Grade 10.

Procedure. The task was Question 10 in a 45-minute written survey with 15 questions, many with multiple parts (see Watson, Kelly, Callingham, & Shaughnessy, 2003, for the full survey). It was the only question based on a bar graph or a graph from a newspaper. The instruction, "Comment on any unusual features of the graphs," was intended to motivate students to consider various aspects without being so explicit as to influence students' focus. Two large labelled spaces were provided to encourage careful consideration and reflect the plural use of the word "feature," thus

permitting two responses. The task behaved well in a measurement sense (Watson et al., 2003) based on the two different hierarchical codings described below.

Analysis. For the purposes of coding the two responses were treated together. Coding was conducted by a research assistant and the first author following the development of two coding schemes. The first scheme, in Table 1, reflected the appropriateness of responses based on the information in the graph and the steps to Critical Statistical Literacy (CSL) noted earlier. The four coding levels of increasing appropriateness had various subcategories defined to reflect the diversity of responses. The second coding scheme, shown in Table 2, was based on the increasing structural complexity of responses in Appreciation of Variation (VAR). Four levels of response were defined, with one having three subgroupings. As indicated by their definitions these two coding schemes were used to reflect the different possible interpretations of the task based on the twin aims of investigating variation and statistical literacy.

Code	Sub Code	Description of Category for Critical Statistical Literacy (CSL)
0		Inappropriate responses
	0A	No response
	0B	Idiosyncratic/"nothing unusual"
	0C	Inferring from graph: Advice
	0D	Direct graph interpretation, without mentioning anything unusual
	0E	Incorrect graph interpretation of unusual data
1		Partially correct interpretation: Unusual data or graphing
	1A	Very general comments about graphing elements
	1B	Both correct and incorrect interpretations of unusual data
2		Correct graph interpretation: Unusual data or graphing
	2A	Correct but non-specific interpretation of unusual data
	2B	Specific statistical comment about graphing elements
3		In-depth graph analysis: Recognises mistakes
	3A	Identification of a mistake, but error in explanation
	3B	Correct identification of at least one mistake

Table 1: Coding scheme based on Critical Statistical Literacy criteria

Code	Sub Code	Description of Category for Appreciation of Variation (VAR)
0		No acknowledgement of variation
1		Focus on columns
	1A	Focus on a single column
	1B	Comparison across two columns
	1C	Focus on the highest column as "most"
2		Focus on increase in the data over time
3		Acknowledgement of variation

Table 2: Coding scheme based on criteria related to Appreciation of Variation

All responses had two codes associated with them, one for CSL and one for VAR. As an example, the response "That 35 people died from not wearing life jackets, 8 from alcohol" was coded as 2A in the CSL coding scheme for its non-specific

interpretation of the data, and as 1A in the VAR coding scheme for its focus on single columns. The research assistant coded the responses, which were checked by the first author, with inconsistencies decided by discussion (cf., Miles & Huberman, 1994).

RESULTS

The results report students' response categories distinguished by the coding schemes for Critical Statistical Literacy (CSL) and Appreciation of Variation (VAR). Changes in response levels over two years are also reported for some students. Students' full responses have been edited in some cases to show only the salient features.

Responses for the CSL coding scheme

As shown in Table 3, a large percent of students in both Grades 7 (44%) and 9 (40%) did not respond at all to the task (Category 0A). Typical responses in Category 0B indicated nothing unusual or were idiosyncratic, such as "They all look okay to me." Some students focused on giving advice based on the information in the graphs (Category 0C), rather than something unusual; for example, "People should wear life jackets." Category 0D responses commented on something in the graph but without focusing on anything unusual, such as "The graphs show us that boats are just as dangerous as cars are." Finally, Category 0E contained responses that identified as unusual something that would not be considered unusual in a statistical sense or that was not based on information in the graph, as seen in "Hardly anyone wore life jackets in 99" or "Less people died by not wearing life jackets."

Code	0					1		2		3		
Sub Code	0A	0B	0C	0D	0E	1A	1B	2A	2B	3A	3B	Total
Grade 7	68	7	5	3	5	2	3	53	6	2	2	
Subtotals	88 (56)					5 (3)		59 (38)		4 (3)		156
Grade 9	67	9	0	4	0	2	8	61	13	2	2	
Subtotals	80 (48)					10 (6)		74 (44)		4 (2)		168

Table 3: CSL categories: number (percent) for each grade

At Level 1, responses were partially correct and addressed unusual data or the format of the graph. Category 1A responses made very general or vague statements, such as "They're all different graphs. They're [sic] all got different meanings." Category 1B responses included both correct and incorrect interpretations. One student wrote "Most people drowned in 1999. A lot of people were tanked [drunk]."

Responses at Level 2 reflected what students considered unusual features of the data or graphs but which were not related to the errors therein. In category 2A were non-specific comments about the unusual nature of the data. Examples include "The number of deaths has risen over the years." The other subcategory of Level 2 (2B) was much smaller, consisting of at least one comment on something unusual about the graphs themselves; for instance, "The way they're set out. They don't have anything telling you what the Y and X axes are."

Of the Level 3 responses that found mistakes, the first group (3A) made errors in reporting these, e.g., "Well on graph 1 it says there is a total of 46 but I counted and it has only got 38." In the final group (3B), responses focused correctly on the discrepancies in the graphs, such as "The first graph has a mistake, the 6 is on 2."

Responses for the VAR coding scheme

Level 0 responses for the VAR coding summarized in Table 4 included both non-responses and responses that had no comments that would indicate a student had considered change or variation in the graphs. Many of the latter, such as "The first one says total: 46, but the graph shows 50 people" may have been placed at much higher levels in the CSL coding.

Code	0	1A	1B	1C	2	3	Total
Grade 7	89 (57)	40 (26)	3 (2)	14 (9)	4 (3)	6 (4)	156
Grade 9	97 (58)	26 (15)	13 (8)	22 (13)	5 (3)	5 (3)	168

Table 4: Variation categories: numbers (percent) for each grade

At Level 1 there were three subcategories focusing on columns. In the first (1A), attention was given to a single column. One student wrote "35 people weren't wearing a life jacket. Tonnes of people keeled over [died] in '99." Category 1B responses considered two columns: for example, "From '87 the deaths have shot up from 6 to 12". In the third group (1C), responses focused on the highest column as "most", as exemplified in "There were more deaths in '99 than any others."

Level 2 responses recognised increases in the data over time, and included "The number of deaths has risen over the years." Level 3 responses made a comment relating to the variation, such as "Most of the people who die were spread out but there was a major increase in '99."

Change in response categories over two years

Table 5 shows the categories of responses related to CSL for the subgroup who completed the survey two years later. There was some improvement, with fewer students responding at Level 0. The Grade 7/9 students performed better than the original Grade 9 cohort, due to an increased number of Level 1 responses. Both groups showed a small increase in the number of Level 3 responses, and the Grade 9/11 students also increased their number of Level 2 responses.

For students with complete data across both years, 43% of Grade 7/9 and 56% of Grade 9/11 responded at the same level (0 to 3), whereas 39% of Grade 7/9 and 26% of Grade 9/11 improved. One Grade 9 student who had originally written "Too many recreational deaths. Not many alcohol related deaths" (2A) gave the following Category 0C response two years later: "People wearing a life jacket, in sheltered waters in a boat under 6m should always have a life jacket." A Grade 7 student who had originally identified a discrepancy in the graph (3B) later focused only on specific aspects of the data, saying "a lot of people die in 1999" (2B). One Grade 7

student who initially focused on specific data elements in writing "Not many people had life jackets. Tons of people drunk" (1B), identified the error in the graph totals in his longitudinal response (3B). A Grade 9 student who gave no response at all in the initial survey later engaged in the task with a Category 1B response: "99 results are extremely high all of a sudden. Alcohol was the cause for almost half."

Code	0					1		2		3		
Sub Code	0A	0B	0C	0D	0E	1A	1B	2A	2B	3A	3B	Total
Grade 7/9	27	1	2	5	6	0	29	50	5	7	5	
Subtotals	41 (30)					29 (21)		55 (40)		12 (9)		137
Grade 9/11	7	5	1	3	2	0	7	26	7	3	2	
Subtotals	18 (28)					7 (11)		33 (52)		6 (10)		64

Table 5: CSL categories — Longitudinal survey (cf. Table 3)

Table 6 reports on similar data but for the VAR coding. Again there was some improvement over the two-year period, and once more the Grade 7/9 students performed better than their earlier Grade 9 counterparts. Most of the change was due to increased numbers in Category 1C and Level 2. Responses clearly articulating variation across the graphs (Level 3) were still rare; however some of them indicated a significant change for some individuals. One Grade 7 student's response in the first survey—"I can't see anything in them. I don't know what the 87 to 99 means"—was coded low on both CSL and VAR, but the student's response in the second survey recognized that the range is big (Level 3). Another student initially could identify particular data elements, and then two years later had a more holistic view, writing "I think that the graph 'Recreational boating deaths' was fairly inconsistent throughout the years and had a sudden jump at the end (99)" (Level 3).

Code	0	1A	1B	1C	2	3	Total
Grade 7/9	47 (34)	30 (22)	15 (11)	20 (15)	17 (12)	8 (6)	137
Grade 9/11	23 (36)	10 (16)	2 (3)	14 (22)	11 (17)	4 (6)	64

Table 6: VAR categories — Longitudinal survey (cf. Table 4)

For students with complete data across both years, 43% of Grade 7/9 and 44% of Grade 9/11 responded at the same level (0 to 3), whereas 45% of Grade 7/9 and 32% of Grade 9/11 improved. There were some who declined quite markedly from the first survey to the second. This includes students who recognized variation initially but who, in the second survey, focused on single columns (e.g., "In 99 they went up but they should have been down because of the new technology and laws") or claimed not to see any "unusual features" at all. Considering both CSL and VAR, no student changed from Level 3 on one to Level 3 on the other. Overall, 14% performed at Level 3 on at least one (one student did so on both).

DISCUSSION

The importance of variation and critical statistical literacy

If the task had been more specific—targeting a particular aspect of either variation or statistical literacy—larger numbers of students may have given higher level responses. It is important to recognize, however, that in articles like that used as a basis for the task and in much of the data encountered daily in the real world, the data come with no questions at all. Accompanying reports often present the writer's interpretation, which may be biased or incorrect. Moreover, as seen here, the actual data as presented come with no guarantees of correctness. Given this, the results of this study are of concern, since students appear to lack strategies for searching for the "unusual". They rarely query the data or examine the data in a holistic way. For the students in this study, the kind of critical thinking suggested by Gal (2002) and explicated as the third step of Watson's 1997 hierarchy seems to be uncommon.

For the CSL coding, 40% to 50% of the students could make generally meaningful comments about what the graphs were showing or, to a lesser extent, could identify technical shortcomings in the graphical presentation. In contrast, a similar number either made no comment at all, or could attempt only vague descriptions. As seen in Tables 3 and 5, less than 10% of the students appeared to check the data in any way for consistency. Presumably the remaining students took the data at face value.

Performance in relation to the VAR coding was similarly disappointing, with well over half of the students in the first survey not considering the variation in the graphs as something that might be regarded as unusual. If variation was acknowledged at all, in most cases it was because students identified particular values, notably extreme values. Students rarely identified—or, at least, commented on—trends or variation in data. There seems to be an inability (or unwillingness) to step back from individual data points in the graphs to make meaning on a larger scale. Those who commented on variation generally gave enough discussion in their response to warrant a Level 2 classification on the CSL coding, but those few who identified errors (at Level 3 on the CSL scale) usually were at only Level 1 on the VAR coding.

Implications for teachers

The results suggest that there has been a lack of attention to variation and statistical literacy with respect to graphs in the media. It is suspected that students are given opportunities to construct graphs based on data, to comment on the technical presentation of existing graphs, and to read off values from graphs and tables, but that critical evaluation and higher level analysis are rarely explicitly fostered. Examples such as the boating deaths graphs used here are not rare in the media, but students need activities that help them to focus on whether the data are internally consistent, whether there are unusual values, whether there are any trends in the data, and how the data vary. Most importantly, they need to make meaning from the data. Teachers can model the kinds of questions that students could and should ask when examining data. A discussion might proceed along the following teacher-directed sequence, depending on students' intermediate responses. "What story do the graphs tell? … Is there anything about them that you consider unusual? … Are there any mistakes in the graphs? … How might you tell the story better?"

Limitations and directions for future research

It should be noted that some students may have lacked motivation to respond seriously to the task, as indicated by some terse or coarsely expressed responses. The authors are also aware that some students may not have been able to express themselves clearly or in detail when completing a written survey, especially given its length. The use of an interview setting is likely to provide richer data. It would also be interesting to use the task with adults, because their experience of real world data since leaving school may affect the kinds of things they perceive as unusual. Finally, can explicit instruction in looking for the unusual in data help students make this their usual approach to examining statistical material?

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PROOFS THROUGH EXPLORATION IN DYNAMIC GEOMETRY ENVIRONMENTS

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The recent development of powerful new technologies such as dynamic geometry softwares (DGS) with drag capability has made possible the continuous variation of geometric configurations and allows one to quickly and easily investigate whether particular conjectures are true or not. Because of the inductive nature of the DGS, the experimental-theoretical gap that exists in the acquisition and justification of geometrical knowledge becomes an important pedagogical concern. In this article we discuss the implications of the development of this new software for the teaching of proof and making proof meaningful to students. We describe how three prospective primary school teachers explored problems in geometry and how their constructions and conjectures led them "see" proofs in DGS.

INTRODUCTION

DGS has revitalized the teaching of geometry in many countries and has made necessary a radical change to the teaching of proof (de Villiers, 1996). One of the most welcome facilities of dynamic geometry is its potential to encourage students' "research" in geometry. In such a research-type approach, students are inducted into theorem acquisition and deductive proof. Specifically, students can experiment through different dragging modalities on geometrical objects they construct, and consequently infer properties, generalities, or theorems. Because of the inductive nature of DGS, the experimental and theoretical gap that exists in the acquisition and justification of geometrical knowledge becomes an important pedagogical and epistemological issue. In this paper, we discuss the pedagogical aspects of introducing DGS into the teaching of geometrical proofs and we provide some indications of how DGS can be used to offer insight and understanding of proofs through investigation and experimentation.

THEORETICAL FRAMEWORK AND PURPOSE OF THE STUDY

The Gap Between Proof and Exploration

The exploration of a problem is by its nature empirical, and, at a first glance, it seems that it does not fit into the deductive character of geometrical proofs. When the empirical and inductive dimension is to be added to the pedagogical structure that is traditionally rooted in deductive logic, one has to combine these two seemingly opposite perspectives. The problem of combining inductive exploration with the deductive structure of geometrical proofs has been the subject of a number of research studies (Mariotti, 2000). The traditional teaching emphasizing that a mathematical statement is true if it can be proved, led students distinguish proof from exploratory activities. However, de Villiers (1996) and (Hanna, 2000) indicated that

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in actual mathematical research, mathematicians have to first convince themselves that a mathematical statement is true and then move to a formal proof. It is the conviction that something is true that drives us to seek a proof. In DGS, students can easily be convinced of the general validity of a conjecture by seeing its truth displayed on the screen while geometrical objects undergo continuous transformations (de Villiers, 1996, 2003).

A number of researchers showed that the passage from "exploratory" geometry to the deductive geometry is neither simple nor spontaneous. Hoyles and Healy (1999) indicated that exploration of geometrical concepts in a DGS environment could motivate students to explain their empirical conjectures using formal proof. They found that DGS helped students to define and identify geometrical properties and the dependencies between them, but when students worked on proofs, they abandoned the computer constructions. The latter leads to the argument that DGS may be useful only in helping students understand problems in geometry but it does not contribute to the development of their abilities in proofs, reinforcing the idea that there exists a gap between dynamic geometry and proof. This may also be the reason that some educators and researchers expressed their concerns and worries that DGS could lead to the "further dilution of the role of proof in the high school geometry" (Chazan, 1993, p. 359). However, the main discussion of recent research, and the main purpose of the present study were to find out ways of effectively utilized DGS to introduce proof as a meaningful activity to students. This can be achieved by reconceptualizing the functions of proofs.

The Functions of Proof

Proof performs a wide range of functions in mathematical practice, which are reflected to some extent in the mathematics curricula. The NCTM Standards (2000) emphasized in a special section on reasoning and proof, the investigations, conjectures, evaluation of arguments and the use of various methods of proofs. From NCTM's document it is assumed that proof is not only understood in the traditional rigid and absolute way, but it also embraces many other functions. Hanna (2000), based on recent research on proof, provided a list of the functions of proof and proving: verification, explanation, systematization, discovery, communication, construction, exploration, and incorporation. She also considered verification and explanation as the fundamental functions of proofs, because they comprise the product of the long historical development of mathematical thought. Verification refers to the truth of a statement while explanation provides insight into why this statement is true.

Traditionally, the function of proof has been seen almost exclusively in terms of the verification of the correctness of mathematical statement. The idea is that proof is used mainly to remove either personal doubt and/or those of others; an idea which has one-sidely dominated teaching practice and most of the research on the teaching of proof. However, de Villiers (2003) proposed other important functions such as

explanation, discovery, intellectual challenge and systematization, which in some situations are of greater importance to mathematicians than that of mere verification.

Edwards (1997) defined the term "conceptual territory before proof" by indicating that conjecturing, verification, exploration and explanation constitute the necessary elements that precede formal proofs. The conceptual territory provides the arena for the construction of intuitive ideas that may subsequently be tested and confirmed through formal methods, and it is the basis for a richer understanding of a proof. This approach reflects the "quasi-empirical" view of mathematics in which understanding proceeds from students' own conjectures and verifications to formal proofs (Chazan, 1993). Simpson (1995) differentiated between "proof through logic", which emphasized the deductive nature of proof, and "proof through reasoning", which involved most of the functions of proofs as were listed by Hanna (2000). Proof through reasoning is accessible to a greater proportion of students, because it is closer to the learning style of students, it makes mathematics more useful and enjoyable, and it reflects the quasi-empirical view of mathematics and the process adopted by mathematicians when they invent mathematics (Simpson, 1995).

The functions of proofs and DGS

The availability in the classroom of DGS gave a new impetus on the teaching of geometry based on students' investigations and explorations. This does not mean that proof is replaced by explorations. On the contrary, exploration is not inconsistent with the view of mathematics as an analytic science or with the central role of proof. Polya (1957) emphasized the connection between deductive reasoning with exploration. He pointed out that solving a problem amounts to finding the connection between the data and the unknown, and to do it, one must use a kind of reasoning based on deduction. In the DGS environment students acquire understanding through verifying their conjectures and in turn this understanding solicits further curiosity to explain why a particular result is true. Students working in the DGS environment are able to produce numerous corresponding configurations easily and rapidly, and thereby they have no need for further conviction/verification (Holzl, 2001). Although students may exhibit no further need for conviction in such situations, it is important for teachers to challenge them by asking why they think a particular result is true (De Villiers, 2003, 1996). Students quickly admit that inductive verification merely confirms and the "why" questions urge them to view deductive arguments as an attempt for explanation, rather than verification (Holzl, 2001). Thus, the challenge of educators is to convey clearly to the students the interplay of deduction and experimentation and the relationship between mathematics and the real world (Hanna, 2000).

THE STUDY

This article presents an account of the thinking exhibited by three prospective primary school teachers while attempting to answer proof problems. It is conjectured that DGS provides an appropriate context where the significance of proof may be unforcefully recognized. To this end, the development of "appropriate" tasks was necessary. By "appropriate" we mean tasks where proof may be providing insight-illumination into why a result, which can be seen on the screen, is true. Open-ended problems seemed as more "appropriate" for two main reasons: (a) statements are short and do not suggest any particular solution methods, and (b) questions are different from traditional closed expressions such as "prove that …", which present students with an already established result (Jones, 2000). Open-ended problems give students the opportunity to engage in a process, which utilizes a whole range of proof functions: exploring a situation, making conjectures, validating conjectures and proving them. The implicit assumption is that during this process students will not have to prove something that they are presented with and do not understand, but something that they have discovered, validated and is meaningful to them. The participants in this study have been asked to work on the following open-ended problem suggested by de Villiers (1996):

Problem: Construct a kite and connect the midpoints of the adjacent sides to form an inscribed quadrilateral. What do you observe in regard to this inscribed quadrilateral? Write down your conjecture. Can you explain why your conjecture is true? Change your kite into a concave kite. Does your conjecture still hold?

After the exploration of this problem, students were engaged in proving similar geometrical theorems. The aim of these additional problems was for students to utilize the proving process in systematizing and generalizing their results.

Students' Proofs

Three prospective primary school teachers with prior experience in dynamic geometry participated in this study. These students had attended a course on the integration of computers in elementary school mathematics, and thus they had a basic understanding of Sketchpad's drawing, menus, and construction features.

Interviewees participated on a voluntary basis and were interviewed while working on the problem. The interviews were conducted in the mathematics laboratory equipped with computers loaded with the Greek version of the Geometer's Sketchpad. The setting was informal with students being able to analyze and build geometric constructions that they thought would help them solve the problems without any time constrains being set. Unstructured interviews were used to collect the data.

In the following, we analyze students' strategies and try to underline the different aspects and functions of proof. The discussion of students' solutions to both problems

is organized around three phases: (a) the phase before proof, (b) the proof phase, and (c) the phase of intellectual challenge of extending proof to similar problems.

The phase before proof

At this phase students explored the problem through constructing the kite and rearranging the constructed figure by dragging it in different directions. This exploration led students to form their own conjectures about the solution of the problem by visualizing the transformations that resulted by the dragging facilities of the software.

Figure 1 shows the way in which students constructed the kite and consequently the inscribed quadrilateral. Two of the students constructed the kite using the property of perpendicularity of its diagonals (see Figure 1a), while the third one used the property of equal adjacent sides by firstly constructing a triangle and then reflecting it on one of its sides (see Figure 1b). All students managed to find the midpoints of the adjacent sides and connected them with line segments using the appropriate functions provided by the software. They conjectured that the inscribed quadrilateral might be a rectangle and confirmed their conjecture by dragging the vertices of the kite to new positions. Students also realized that their conjectures hold also in the case of the concave kites. All the students evaluated their mathematical conjectures not only visually but also numerically by measuring the sides and angles of the inscribed quadrilateral, confirming that it was a rectangle, and thus verified their conjecture. It is also important to note that these students used the measuring tools for slope to show that the opposite sides of the inscribed shape were parallel. Furthermore, they noticed that the diagonals of the kite were also parallel to the sides of the inscribed shape.

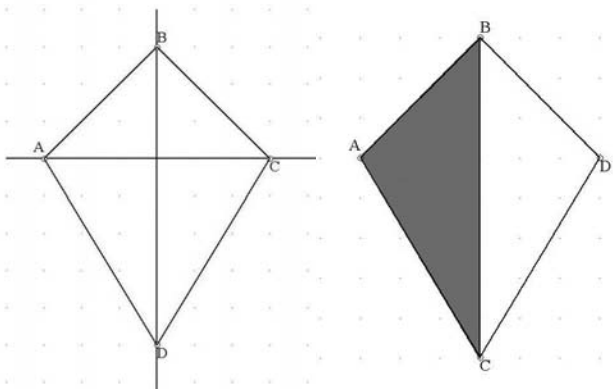

Figure 1a Figure 1b

Figure 1: The construction of kite

The proof phase

The exploration of the problem as it was done in the "phase before proof" led students to become convinced about the validity of their conjecture. This conviction was achieved solely by the use of the dynamic geometry environment. During the "proof phase" the role of proof is not to convince or remove individual or social doubt about a proposition but primarily to find ways to explain why a certain result that can be seen on the screen is true (Jones, 2000). One of the students in this study showed no further need for conviction that the inscribed quadrilateral was a rectangle, while the other two students felt the need to explain why they thought this particular result was true. These two students admitted that the inductive verification they provided for the mathematical statement was not satisfactory in the sense that the inductive process was not a consequence of other familiar results. Furthermore, they proceeded to view a deductive argument as an attempt for explanation, rather than for verification.

At this phase, the DGS enabled students to pass from "exploratory" geometry to deductive geometry, bridging in this way the gap between dynamic geometry and proof. Specifically, the two students, who successfully solved the problem, based on the measurements they made earlier on in the exploration phase (the pre-proof phase), defined and identified the geometrical properties and the dependencies between them, and provided a deductive proof of the problem. In fact, they realized from their measurements that EF, and HG are equal to ½ AC (see Figure 2). This directed them in what they needed to look for in their geometry books, where they found the respective theorem. Based on this property they showed that EF is equal and parallel to HG as well as EH is equal and parallel to FG, and therefore EFGH is a parallelogram. The next step was to prove that the parallelogram was a rectangle, i.e., one at least of the angles of the parallelogram was a right angle. Based on the property of the perpendicularity of the diagonals of the kite, students observed that since BD $\perp$ AC, then EF $\perp$ EH, which implies that EFGH is a rectangle. (The dragging facility of the software enabled students to conceive that their explanations hold even in the case of concave kites).

The phase of intellectual challenge of extending proof to similar problems

In this phase we discussed two categories of problems: (a) problems that have a similar context to the kite problem, and (b) problems that require the same type of reasoning. The purpose of the problems in the first category was to help students generalize their finding from the kite problem to quadrilaterals of various types. To this end, the three students tried to systematize their experimentations by investigating first the more familiar quadrilaterals such as parallelograms, rectangles, rhombuses, squares, rectangles and then they proved, using the same explanations as they did in the kite problem, that in any quadrilateral the shape resulting from the midpoints of its sides is always a parallelogram. The purpose of the second category

of problems was to ensure that students could easily transfer the proving process to problems with different structure.

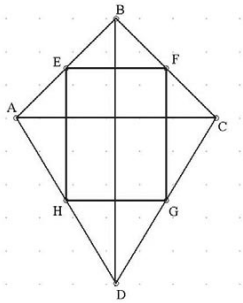

Figure 2: The proof that the inscribed quadrilateral is rectangle

CONCLUSIONS

In this paper we tried to show some of the ways in which DGS can provide not only data to confirm or reject a conjecture, but ideas that can lead to a proof. To this end, the results of the study were presented in three phases: the phase preceding proof, the proof phase, and the phase of intellectual challenge of extending proof to similar problems.

The phase preceding proof is quite necessary for students to understand the problem based on their own intellectual efforts. In the kite problem students encompassed their informal reasoning and argumentation that came into play when students worked from their own investigations (Edwards, 1997). To construct the kite, which was a challenge by itself, students first investigated its properties and then tried to apply them on the computer screen. The graphing and validating capabilities of DGS enabled students to explore the problem and make mathematical conjectures. In turn, students checked specific cases of kites, using the dragging facility of the software, to see if their conjecture holds true, i.e., the shape formed by connecting the midpoints of adjacent sides of a kite is always a parallelogram. In other words, the phase preceding proof helped students to build up empirical evidence for the plausibility of their conjectures.

A number of research studies indicated that engaging students in the phase preceding proof did not necessarily lead them to an awareness of the need for proof (Chazan, 1993; Edwards, 1997). On the contrary, in the present study, we found that DGS and appropriate questions prompted or motivated students to seek justifications for their conjectures. Two of the three students in this study justified their conjectures for the kite problem based on the screen outputs. In addition, students in the study did not support that their experiments and measurements were sufficient to support a geometrical statement. Measurements functioned as a means for finding explanations and a means for gathering information for justifying their results. The relations

between the measurements in conjunction with the invariant properties of the shapes functioned as students' hints into explaining their conjectures. Measurements also provided students with specific examples that formed the ground for further conjectures and generalizations. It is in this area that the computer contributed to students' attempts toward proof and bridging the gap between inductive explorations and deductive reasoning. This became more apparent during the phase of intellectual challenge of extending proof to similar problems. During the last phase, which was not adequately presented due to space limitations, students felt a strong desire for explaining their conjectures and understanding how one conclusion is a consequence of other familiar ideas, results or theorems. Students found it quite satisfactory to view a deductive argument as an attempt for explanation rather than for verification (de Villiers, 2003).

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ESTABLISHING A PROESSIONAL LEARNING COMMUNITY AMONG MIDDLE SCHOOL MATHEMATICS TEACHERS

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The paper examines how community was established in a professional development institute that focused on algebra content knowledge for middle school mathematics teachers. This qualitative study was framed within a situative perspective. We analyzed multiple data sources to identify the ways in which community was established. Results indicate that giving tasks that provided access to all participants on the first day allowed active participation from all participants and characteristics of community emerged. Characteristics that were evidenced in triangulated data indicate that explaining and clarifying ideas, building off of others' ideas, admitting weaknesses, giving praise to others, and laughing were indicators that community was being established.

BACKGROUND AND FOCUS STATEMENT

Professional development models are receiving renewed attention in mathematics education. Researchers are examining a variety of methods to identify characteristics of models that provide promise for improving classroom teaching and student achievement. Research suggests that one feature of successful professional development models is the ability to create community (Cobb, McClain, Lamberg, and Dean, 2003; Franke & Kazemi, 2001; Grossman, Wineburg, & Woolworth, 2001; Stein, Silver, & Smith, 1998).

Learning within a community of teachers is a simple idea yet establishing a successful community that results in teacher change and student achievement is a complex endeavor. First and foremost, the teachers need to share in the commitment to intellectual development and refinements in practice (Elmore, Peterson, & McCartney, 1996). Other features of professional development programs that support community development include: creating a safe environment for teachers to grapple with difficult content and pedagogical issues, developing sustained relationships among teachers in the community, encouraging participants to listen carefully to each others ideas and perspectives, equally distributing social and intellectual work within the community, and fostering a commitment to helping others within the group learn and develop both intellectually and in their teaching practice (Wenger, 1996; Grossman, Wineburg, & Woolworth, 2001.)

The most developed models of teacher community within professional development programs originated in elementary school mathematics (Carpenter, Fennema, & Franke, 1996; Schifter, 1996; Franke & Kazemi, 2001). These models of professional development focus on in-depth understanding of the elementary curriculum from a

student thinking and learning perspective. At the heart of these models is the assumption that teachers teach concepts that they themselves have not mastered. Teacher learning is defined as understanding elementary mathematical concepts and curriculum. Elementary teachers often do not possess extensive mathematical knowledge, and one of the reasons for community is to mitigate teachers' negative affect around difficult subject matter (Schifter, 1996). Teaching communities within professional development models differ between grade level and subject matter (Grossman, Wineburg, & Woolworth, 2001). There is a difference between professional development communities in the elementary school in that elementary school teachers are not expected to be subject area experts whereas in high school communities many teachers have degrees in mathematics and sometimes advanced degrees. However, professional development communities in the middle school are unique in that that they are made up of high school licensed teachers as well as elementary school licensed teachers. A middle school mathematics community makes a unique community in that it adds different strengths and weaknesses to the professional development. This paper describes how community evolved within a summer institute for middle school teachers on conceptualizing algebra.

THEORETICAL FRAMEWORK

We drew upon a situative perspective to design both the professional development institute and the research investigation (Greeno, Collins, & Resnick, 1996; Putnam & Borko, 2000). From the situative perspective, a critical aspect of professional development is the development of community. We draw on the work of Lave, Wenger, and Grossman, Wineburg, & Woolworth to define community. Lave (1996) defines community of practice as relations across people, and activity over time and in relation with other communities of practice. Grossman, Wineburg, and Woolworth capture the notion of professional teacher community by indicating necessary speech and action enacted within the group. A professional teacher community is characterized by: [or "develops through" 1) the formation of group identity and norms of interaction, 2) the navigation of differences among group members, 3) negotiating the essential tensions between the goals of improving professional practice and fostering intellectual development, and 4) communal responsibility for individual growth. (Grossman, Wineburg, & Woolworth, 2001). Fundamental indicators of learning within the situative perspective are identifying changes in participation in the social practices of a community (Greeno, 2003; Lave, 1996). Therefore, professional development created with community as a central characteristic create an environment that considers participation, social negotiation, and collective learning. Social negotiation including the regulation of social interactions and group norms is an ongoing practice. Originally a few key individuals may do most of this regulation however roles in leadership will shift overtime.

From the situative perspective, the evolution of teacher professional development communities can be documented by observations of changes in leadership and shifts

in participation (Rogoff, 1997). Indicators of group equity and maturation can be identified by the degree to which discussion brokering is distributed among individuals and the degree to which it is shared rather than monopolized by one or two people. Members of the teacher community must believe in the right to express themselves honestly without fear of censure (Grossman, Wineburg, & Woolworth, 2001). Documenting how this evolution takes place is different from professional development to professional development. Yet, as members transform their role within the community the person they are becoming crucially and fundamentally shapes what they know (Lave, 1996, p. 157) and indicators of growth can be identified. Genuine communities make demands on their members as membership comes with responsibilities. These demands can also be outlined as markers of maturation. More specifically, in a teacher community-a core responsibility is to help other teachers learn by encouraging them to contribute to large group discussion, pressing others to clarify their thoughts, eliciting the ideas of others, and providing resources for others' learning.

METHOD

The Professional Development Institute

The summer algebra course was part of the "Supporting the Transition from Arithmetic to Algebraic Reasoning" (STAAR) project. STAAR is an NSF-funded, 5-year project, conducted collaboratively between 3 major universities. The aims of the project are to study algebra teaching and learning at the middle school level, focusing both on students and teachers. The general scope of the summer algebra course was jointly developed by members of the STAAR team and based on two years work from three tiers of the project. The course was grounded in emerging theories about how students develop algebraic reasoning identified by Tier 1, how teachers teach algebraic concepts identified by Tier 2 and the professional development described here was to help teachers foster the transition from arithmetic to algebra designed by Tier 3. There was a general consensus among the team that middle school teachers might benefit from extended learning opportunities centered upon the teaching of algebraic reasoning. The two-week STAAR summer algebra course was held in July 2003 at a university campus setting. The three-credit graduate level course was offered through the Continuing Education program in the University's School of Education. According to Putnam & Borko (2000), such a setting appears to be "particularly powerful… for teachers to develop new relationships to subject matter and new insights about individual students' learning" (pg. 7).

Course Goals

Increasing teachers' content knowledge was the central goal of the course. The aim was to challenge teachers' own content knowledge as they engaged in rich explorations of many of the algebraic concepts that they are likely to teach in their own classrooms. Creating a teacher community or network was another goal of the summer course. Developing a sense of community among the teachers in the course was deemed very important. The course encouraged teachers to work together by

seating teachers in small groups, assigning mathematical problems and encouraging teachers to work on them together.

A third goal was to have teachers experience learning in a classroom based on "reform" ideals. A fourth goal was to increase teachers' awareness of students' algebraic thinking by examining student work, discussing student thinking, and reading current literature. A fifth goal was to influence teachers' beliefs about algebra and pedagogy. In particular, the course was designed to help participating teachers see the value in developing algebraic reasoning skills through problem solving, group work, sharing a variety of solution methods, etc. This presentation focuses on the second goal, tracing how the professional teaching community evolved within the algebra summer course.

Participants

Sixteen mathematics teachers participated in the course. They were all inservice teachers from a variety of school districts in the state, mostly teaching at the middle school level. Although there was a range of experience among the teachers in the class (from 0-15 years), the majority had relatively little experience teaching middle school algebra. The course was team taught by two mathematics educators. One of the instructors had mathematics teaching experience at the middle school level while the other taught at the secondary level. Both also had experience teaching university courses and mathematics professional development courses.

Data Collection

An extensive set of data were collected both to describe the teaching and learning that occurred within the context of the summer course, and to track changes in the participants' knowledge and beliefs. Two video cameras were used throughout the course to document the activities of the instructors and the students. During whole-class activities, one camera was focused on the instructor(s) and the other on the students. When students worked in small groups, the cameras were trained on separate groups of students. In addition, extensive daily notes were kept by several members of the research team. Multiple measures were used to assess teachers' mathematical abilities and beliefs before and after the course. These measures included written mathematics assessments, face-to-face (or telephone) interviews about their beliefs regarding algebra teaching and learning, and a written statement about their mathematics experiences. The participants also kept extensive documentation of their work and reflections during the course. The course instructors were interviewed on a daily basis about their reflections on the class sessions. They also kept records of all instructional plans, handouts, and assignments.

The conversations that occurred throughout the professional development course, captured on videotape, are a main source of data to document how community evolved. In addition, field notes, interviews with both of the instructors and teachers, teacher daily reflections and instructor interviews provide important data for triangulation, confirming and disconfirming evidence. If a group grows toward

community you should be able to hear it and see it in the venues (PD, online) in which they met. Claims should be supported by evidence from the interactions of the members.

Analysis

The coding framework implemented in this study uses two categories of codes. The first category includes 13 high inference analytic codes, for which the unit of analysis was whole discourse events. The second category includes 14 low inference analytic codes; which are applied to smaller chunks of data examined line by line (See Table 1.). As an initial step in data analysis, one researcher viewed the entire set of video recordings, created a chronological record of activities within the professional development institute along with a brief summary of each activity. At the same time, she identified activities during which issues related to the evolution of community were particularly evident. A second researcher analyzed these sections using the coding framework. A third researcher went through the data sets to achieve inter rater reliability in the coding of the data. Interrater reliability was accepted data was coded with 90% agreement. When coding of data was complete, researchers went through the data set and clustered codes. Themes were determined from the clustered data set. Again we went through the data set to find confirming and disconfirming evidence using triangulated data for the themes that emerged.

Code	High Inference	Code	Low Inference
TR	Sharing specific tools, representations, and artifacts	SW	Sharing a weakness or misunderstanding
SS	Shared stories, inside jokes, laughter-	SP	Sharing ideas and ways of thinking
RT	Reoccurring themes in language or in solving problems	CI	Challenging each others ideas
IC	Participants performed both individually and collectively to make sense of problems	CT	Instructor(s) gave tasks that required cooperative skills

Table 1: Sample of Select Analytic codes

RESULTS AND CONCLUSIONS

Preliminary results suggest the following themes emerged from the data set that help to characterize the evolution of community within the summer algebra institute. First, the grouping of participants was a planned strategy for community building. Instructors specifically grouped participants by personality to stimulate community development over the course of the institute. For example, the instructors' goals were to place participants that did not have prior experiences with each other together so that students made new relationships on the first day. These decisions were based on

prior relationships with the participants and knowledge from institute interviews (Instructor Interview Day 1, July, 2003).

Second, it appeared that using tasks that provided access to all participants was important in the co-involvement of the whole group as they solved problems. Problems used on the first day appeared to be puzzle like tasks with multiple levels of access. All participants could solve the "sidewalk problem." This task had students cut a rectangular section of a sidewalk and find the minimum and maximum pieces that could be made. All participants could make representations of the different problems involved. Conversations focused around finding the minimum and maximum in which all group members participated, generalizing results, and writing algebraic expressions to represent the problem. This problem led participants to clarify and explain what they knew and did not know, be persistence in problem solving, admit when they did not understand the thinking or the content that others used, giving praise, and laughing. The sidewalk problem is a series of problems that was used most of the first day. The students worked in small groups where they individually solved the problem as well as collaboratively. Intermittently they would share parts with the whole class before they moved forward. Data analysis indicates that clarifying and explaining, building off of each others ideas, persistence, admitting weaknesses and laughing together were all characteristics that appear to be the ways in which community initially began to evolve. The following excerpt provides an example of one group presenting their results near the end of the day.

[Ken, the reporter, gets up to come to the overhead Mary, Mindy, and Allen get up as well.]

Mary: This is a team effort.

Allen: you might need us.

Kris: You guys don't have to come up.

Mary: Cover that part [of the overhead] up.

Ken: You have seen this before maybe [showing a table they made to represent the problem]. We started with the number of lines, then the minimum number, and the maximum number.

[agreement from teammates]

Ken: [jokingly asks] Can I go on? [directed to his teammates responses]

Mia: This was Ken's idea [pointing to the next column on the chart] before lunch. He noticed that something was going on with the number of intersections. Like how many intersections did you have and the resulting number of pieces.

Allen: This is kind of going off of what the last group introduced that the intersections had to increase by 1, 2, 3, etc. This was the actual number of intersections so these should line up.

Mary: [goes to the screen of the overhead] So this would be the 1, 2, 3, 4, as she points to different intersections that were aligned with the previous groups chart. But also, Mia and I

were like these numbers look familiar and Mia remembered they were all triangular numbers and I was like whatever but after she showed me why I was okay I get it.

Ken: This brings up for those of us who paid attention during high school math. [sarcastic saying he did not know triangular numbers previously.] [lots of laughing and references made to other summer content courses.] Oh yeah! Pascal's Triangle! Who is Pascal? I thought that was a chip.

Mary: [nodding her head in agreement]

Ken: And then they came up with this thing which is really pretty awesome if you notice these numbers [circles 1, 6, 10, 15] if you look at the number of intersections you will notice they are the same.

Mary: Allen tell them what you came up with at this point.

Allen: The column over here is exactly what the last group said this is where it increases over 1,2,3,4,5,6,-then the next column is the number of intersections so that is where it increases and where this formula came in [pointing to the overhead and explaining the similarities] We tried to figure out how to get from these numbers [number of intersections] to the maximum.

Ken: So we came up with n is the number of lines which is how we all did it. Then n + 1 is the minimum-most of us came up with that. Then we took that [(n + 1)/2] + 1 would be the maximum which is basically what you can do with the triangles somehow.

Mary: Yeah.

Allen: It is the intersections and you are adding to..

Ken: It's still a little foggy to me…

This excerpt provides a window into the first day of the algebra institute. You heard participants explaining and clarifying their ideas, building off of the previous group's presentation, admitted weakness, giving praise, and laughing and having a good time. On the second day, the tasks that the instructors gave and the pedagogical focus encouraged participants to establish and gain deeper trust in the relationships with first a partner and then the other participants as they grappled with their own content knowledge including understandings, misunderstandings, and the intricate underlying relationships between the conceptual ideas of algebra. This led to in-depth dialogues among participants in which characteristics such as clarifying mathematical ideas, making sense of multiple solution strategies, struggling with a difficult problem, and sharing of weakness or misunderstandings were identified more often. Each of these characteristics emerged as themes to help explain how community was established in the professional development institute. This work adds to the literature base on effective ways to establish community within professional development.

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PATTERNS OF PARTICIPATION IN THE MATHEMATICS CLASSROOM

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This paper first postulates the existence of co-constructed patterns of participation in the classroom and then documents one such pattern of participation: Kikan-shido (instruction between desks), both as it is enacted and as it is perceived by the classroom participants. In the course of detailing the use of kikan-shido in three Australian classrooms, this paper addresses the relationship between practice, participation and learning, as exemplified by kikan-shido as a locally-enacted pattern of participation to which teacher and students subscribe and which teacher and student have agency to exploit and to shape, and in which teacher and students, are complicit (co-conspirators). Acceptance of this point has implications for the research designs by which we study the activities occurring in classroom settings.

INTRODUCTION

Eugene Ionescu is reputed to have said, "Only the ephemeral is of lasting value." Social interactions are nothing if not ephemeral and, since it is through social interaction that we experience the world, the understanding of social interactions must underlie any attempts to improve the human condition. Our difficulties in characterizing social interactions for the purpose of theory building are compounded by the fluid and transient nature of the phenomena we seek to describe. Attempts to categorise social behaviour run the risk of sacrificing the dynamism, contextual-dependence and variation that constitute their essential attributes. This poses a challenge both for methodology and for theory. The ephemeral nature of social interactions is something that must be honoured in the methodology but transcended in the analysis.

The juxtaposition, in this paper, of participation, position, role and practice reflects a perceived gradation in the constructs that might be used to model the fluidity of social interaction. Where *role* is taken to have an institutionalised status and to constitute a more lasting categorization (eg teacher or pupil), while *position* is a social artefact constructed through the interactions of a social group smaller than an institution and less enduring in its membership (eg leader or expert). *Participation* is the social mechanism whereby both positions and roles are enacted and the patterns of participation that provide the focus for this paper are the regularities of *practice* engaged in by a social conglomerate – in this case, a teacher and her/his pupils.

Practice also requires explication. Greeno observed that "Methods of instruction are not only instruments for acquiring skills; they also are practices in which students learn to participate" (Greeno, 1997, p. 9). With regard to the learning of mathematics, some

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classroom practices will resemble those of other communities who habitually make use of skills specific to mathematics (the mathematical activities of accountants or surveyors, for example), and some practices will be classroom-specific in the sense of relating to the process of learning (providing particular forms of explanation, asking particular types of questions when in doubt, seeking and offering assistance, and so on). Greeno also made reference to "patterns of participation" developed by students (Greeno, 1997, p. 9). This is a particularly apt phrase, combining the fluidity of participation in a social setting with the implicit regularity of a pattern. If we are to understand what occurs in social settings, it is the patterns of participation that are likely to offer insight. As will be argued, in considering social interactions in the classroom, the teacher must be considered co-participant with the students in any practices of the classroom community.

In this paper, the notion is posited of an individual having constructed a body of practice in which s/he engages regularly and with some consistency, but which is subject to refinement, modification, rejection, and replacement over time. The practice of individuals (a teacher's practice or a learner's practice) is distinguished from 'professional practice' in the sense of established 'legal practice' or 'medical practice.' Such individual practice will be a subset of the practices of the various communities of which each individual has membership and will conform to the affordances and constraints of the settings and situations in which those individuals find themselves.

Like Wenger (1998), this analysis of patterns of participation in classroom settings stresses the multiplicity and overlapping character of communities of practice and the role of the individual in contributing to the practice of a community (the class). Clarke (2001) has discussed the acts of interpretive affiliation, whereby the learners align themselves with various communities of practice and construct their participation and ultimately *their* practice through a customizing process in which their inclinations and capabilities are expressed within the constraints and affordances of the social situation and the overlapping communities that compete for the learner's allegiance and participation. By examining classroom practice over sequences of ten lessons, the Learner's Perspective Study (website: http://www.edfac.unimelb.edu.au/DSME/lps/ and outlined below) provides data on the teacher's and learners' participation in the co-construction of the possible forms of participation through which classroom practice is constituted (cf. Brousseau, 1986).

But co-construction *of practice* and joint participation *in practice* do not connote commonality of purpose among the participants in that (classroom) practice. To some extent both teacher and student share a common interest in advancing the student's learning, but they are not positioned identically within that purpose (cf. Davies & Harré, 1991), and their classroom participation will both confirm these positionings and co-construct them.

METHODOLOGY

This paper reports the analysis of data from sequences of lessons, supplemented by post-lesson video-stimulated interviews, and uses one particular whole class pattern of participation (*kikan-shido* or instruction-between-desks) to illustrate differences between the function of practice, role and position in the context of eighth-grade mathematics classrooms. Data collection was undertaken consistent with the 'complementary accounts' approach discussed in detail elsewhere (Clarke, 1998 and 2001). In the Learner's Perspective Study (LPS), three video cameras documented teacher and learner actions for sequences of at least ten consecutive lessons and this video record is supplemented by post-lesson reconstructive video-stimulated interviews with teacher and students, together with test and questionnaire data and copies of written material produced in class and interview.

> We need to acknowledge the multiple potential meanings of the situations we are studying by deliberately giving voice to many of these meanings through accounts both from participants and from a variety of "readers" of those situations. The implementation of this approach requires the rejection of consensus and convergence as options for the synthesis of these accounts, and instead accords the accounts "complementary" status, subject to the requirement that they be consistent with the data from which they are derived, but not necessarily consistent with each other, since no object or situation, when viewed from different perspectives, necessarily appears the same (Clarke, 2001, p. 1).

Our goal in the analysis of the classroom events documented in the Learner's Perspective Study is the identification of pattern within the complex, interconnected database. As will be argued, the analysis undertaken in this paper must not only identify a 'pattern of participation' within the data, but also demonstrate that the participants themselves gave tacit (or explicit) recognition to that pattern, by verbal reference in interview or classroom utterance, or through their active participation in conforming to the pattern or even contributing to its construction.

THEORETICAL POSITIONING

Previous research, and much of our theorising, has tended to dichotomise teaching and learning as discrete activities sharing a common context. It has been argued that this dichotomisation is a particularly insidious consequence of the constraints that language (and the English language, in particular) imposes on our theorizing (Clarke, 2001). It is a major premise of this paper (and the project of which it is one product) that such dichotomisation misrepresents both teaching and learning and the classroom settings in which these most frequently occur.

The theory of learning on which this paper is grounded is one that starts from the social situation of the individual in interaction with others, but which accords a significant role to the individual's interpretive activity. Particular significance is attached to social interaction, and learning proceeds by the iterative refinement of intersubjective understandings that include social and content-specific (in this instance, mathematical)

meanings, as well as values and modes of collaborative practice. These understandings are enacted as the progressive increase in valued practice, including the appropriate utilisation of technical language. This account of learning invokes a negotiative process that presumes interaction with others. These interactions are predicated on an interpretive affiliation that situates the learner with respect to the values and goals of others in the learning environment (the classroom) and an interpretive characterisation of the other, by which the capabilities, motivations, values and actions of other participants in the classroom are inferred and this characterisation is then iteratively refined through on-going social interaction. Context is also a matter of interpretation and internalization (see Clarke & Helme, 1998). Essential to an understanding of the nature of social activity in classrooms is the co-constructed nature of the practices of these classrooms, and the role of negotiation not as a subordinate activity *through* which classroom practice is constructed but as an essential activity *from* which classroom practice is constituted.

This paper provides evidence of the mutuality of teaching and learning and supports their interpretation as components of a single body of communally constituted practice. We are assisted in this argument by Harré's work on social positioning (Davies & Harré, 1991) as this gives recognition to the mutuality of social practice, where the positioning of an individual carries both rights and responsibilities and is only sustained by mutual compliance. Of course, a position can be contested and negotiation is a constitutive element of classroom practice (see Clarke, 2001).

Lave and Wenger provide a plausible connection between practice and learning, in which the learning is constituted as participation in practice and the mediating mechanism is the situated negotiation of meaning: "Participation is always based on situated negotiation and renegotiation of meaning in the world" (Lave & Wenger, 1991, p. 52). In this view, participation is not the medium by which learning is afforded, it is the thing itself. As such, patterns of participation take on a heightened significance as established forms of practice. Legitimate participation in institutionalised practice is taken to signify learning or the acquisition of knowledge. My focus in this paper is on those patterns of participation that stand in the same regard to the practices of the disciplines of science or economics as the classroom does to the research laboratory or the stock exchange.

KIKAN-SHIDO (INSTRUCTION BETWEEN DESKS)

Japanese teachers possess an extensive vocabulary with which to describe their practice. Among the myriad terms available to them is the term 'kikan-shido,' which means 'instruction-between-desks' in which, while the students are engaged in "practice", either individually or in groups, the teacher walks around the classroom, observing students at work, and may or may not speak or otherwise interact with the students. This activity is a familiar one to teachers in American and Australian classrooms, and to teachers in many other countries as well. As the translation (Instruction Between Desks) makes clear, the Japanese term for this activity focuses on

describing the teacher's actions. If I were to use the English translation as the label for this pattern of participation, I would be maintaining the focus on the teacher's activity, whereas the whole purpose of my argument is to demonstrate the mutuality of teacher and student participation in this activity. So, for the purposes of this discussion, I will use the Japanese term, 'kikan-shido' as a signifier or cipher for a more general conception of the particular activity – one that takes into account the patterns of participation of both teacher and students in the activity designated by 'kikan-shido.'

In this analysis, I examined kikan-shido from several perspectives: Its form as observed on the video record of class activity; its meaning as reconstructed by teacher and students in post-lesson video-stimulated interviews; and its function (intention, action, and interpretation). My intention in describing and discussing this pattern of participation is to examine the legitimacy of the characterisation of kikan-shido as a whole class pattern of participation, and to situate the actions of teacher and learners in relation to this pattern of participation. It will be argued that while engaging in kikan-shido, the teacher and the students participate in actions that are mutually constraining and affording, and that the resultant pattern of participation can only be understood through consideration of the actions of all participants.

In the Australian LPS video data, it is clear that all three teachers made extensive use of "instruction-between-desks" in every lesson, and commonly for extended periods of many minutes. During this time, the Australian teachers monitored the students' current activities and, sometimes, whether or not homework had been completed. While walking around the classroom, the Australian teachers frequently conversed with the students: Questioning, prompting, and generally scaffolding the students' activity. In the lessons analysed in this study, the scaffolding activity was more likely to involve questioning students than simply telling them an answer or a procedure to use.

For the Australian teachers, the activity of "instruction between desks" appeared to have at least three principal functions: (i) monitoring and encouraging current on-task activity, (ii) actively scaffolding this on-task activity, and, sometimes, (iii) monitoring the completion of homework. On many occasions teachers would kneel or sit beside a student (or students) and engage them in conversation about the task they were attempting.

In presenting this analysis, three examples have been chosen to illustrate the diversity of practice evident within the Australian data (constraints of space prevent the complete reproduction of the data here).

Example1: A1-L8 (0:33:47 to 0:36:26) Guided Questioning by Teacher – Non-Routine Task
Example 2: A3-L8 (0:27:59 to 0:31:52) Guided Questioning by Teacher – Routine Task
Example 3: A4-L12 (0:31:18 to 0:33:27) Explicit Teacher Demonstration

In relation to these examples, it is useful to consider some related interview data (attenuated for reasons of space):

Interview with Teacher 1 (T1)

T1 like I get down on my knees a lot and try not to be . . . I don't want my presence to be *overpowering*. I don't want them to think, "Oh she's *over* me just telling me what to do." I don't want to come down on them

T1 Oh . . . this was *terrible* [slow] . . . I-I ar as soon as I started going around oh I felt bad about this . . .

It just sort of . . . was made *very obvious* that I hadn't . . . but that that's also another thing that I do, I do go to see them straight away so they can tell me . . . what they don't understand – that that gives me a much better . . . understanding of whether . . . what I have done up the front is of is of any value at all.

Interview with Student from School 1

S1 It's really good when Mrs T1 comes around to everyone individually . . . it's so if you are not sure about anything . . . you just like . . . she'll come around.

There are four key aspects to Teacher 1's participation in kikan-shido that emerge from the data illustrated by the brief examples above:

- The students' perception of the teacher's commitment to be "there" for "everyone"
- The teacher's deliberate use of physical positioning to minimise any intimidation of the students and, implicitly, to reduce the prominence of the inevitable power difference between teacher and student
- The inadequacy of the student's use of the term "explain" to encompass the teacher's instructional action. Video evidence suggests that the teacher's actions were commonly much less directive or transmissive than is suggested by the term "explain" as used in student interviews.
- The teacher's utilisation of kikan-shido as the means by which to gauge the success of her whole class presentation.

THE CO-CONSTRUCTION OF WHOLE CLASS PATTERNS OF PARTICIPATION

Of major interest for the purposes of this paper is the evidence that kikan-shido was a pattern of participation to which both teacher and students subscribed and which was co-constructed by them. In designating kikan-shido as an example of a "whole class pattern of participation" I need to demonstrate that it had a recurrent form, recognisable to those participating in it. This is not to say that the meanings attributed to the activity by those participating in it were correspondent. The point has already been made that individuals can participate in a practice whilst being positioned differently within it,

and whilst attributing different characteristics to the activity. That is, without being identical, the participants' descriptions of the activity make it clear that they are talking about essentially the same form, but they may attribute quite different functions to that form. The other essential element is the need to demonstrate that all participants can shape the particular body of practice signified by kikan-shido. That is, that the pattern of participation is co-constructed.

Evidence that students contribute to the form taken by a pattern of participation such as kikan-shido can be found in a statement from a seventh-grade student in an earlier study.

Davy	Oh we do muck around (both laugh). When me and Darren we just talk. When we've got our hands up we just talk during so she comes. Then when she comes we get back to work. Or maybe some hot day we're just talking or mucking around, or pushing people around. Something like that . . . 'Cause sometimes I might have me hand up for five minutes. She's right next to me and she goes over the other side of the room. And that's why I start mucking around . . . so I get her attention.

This provides explicit acknowledgement by the student that the teacher's participation in kikan-shido can be manipulated.

The extent to which kikan-shido, as practised in the Australian classrooms analysed in this study, has distinctive cultural or national features is immediately suggested when classrooms in other countries are investigated for evidence of the same practice. From the comparison of sequences of ten lessons, taught by three competent Australian teachers, with matching data sets from countries such as the USA, Hong Kong, Mainland China, The Philippines, and Germany, it appears that, in general, the Australian teachers commit more time to kikan-shido than do the teachers in the other countries.

CONCLUDING REMARKS

In this paper, I have attempted to frame the argument that any theory of classroom practice must conceive of the activities in the classroom as co-constructed. Kikan-shido as it has been reported here is clearly a dance done by teachers and students, where the steps are improvised according to need. The participants in the classroom, teacher and students, are complicit (co-conspirators) in this improvisation. Acceptance of this point has implications for the research designs by which we study the activities occurring in classroom settings.

A corollary of this point is the problematisation of learning and teaching as distinct processes and as disjoint bodies of practice - at least to the extent that this disjunction is applied to classroom settings. The need has been identified elsewhere (Clarke, 2001) for a single term to encompass the conjoint, co-constructed body of practice signified in Russian by *obuchenie*.

But co-construction *of practice* and joint participation *in practice* do not connote commonality of purpose among the participants in that (classroom) practice. Even where all participants recognize and subscribe to a particular pattern of participation (the example used in this paper is *kikan-shido*), they may interpret its function differently. Nonetheless, the study of patterns of participation offers one approach to capturing both the fluidity of social interaction and its regularities.

If we conceive of institutionalised patterns of participation as taking on the status of bodies of practice, then their co-constructed nature has further significance. Rather than progressively increasing the competence of their participation in a culturally or socially pre-determined practice (eg Lave & Wenger, 1991), this conception of the origins of practice accords significant agency (however constrained by institutional or cultural norms) to the participants to shape their particular pattern of participation and thereby to influence the nature of that practice. Wenger's more recent writing (Wenger, 1998) assigns significantly greater agency to the participants in a practice. This analysis has provided some examples of how that agency is enacted.

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YOUNG "WHITE" TEACHERS' PERCEPTIONS OF MATHEMATICS LEARNING OF ABORIGINAL AND NON-ABORIGINAL STUDENTS IN REMOTE COMMUNITIES [1]

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Despite a plethora of writings on Australian Aboriginal education (Craven, 1998; Fanshawe, 1999; LeRoux & Dunn, 1997; Malcolm, 1998; Malin, 1998; Morgan & Slade, 1998; Partington, 1998; Russell, 1999; Stewart, 1999), little has dealt with teacher perceptions of how Indigenous students learning in comparison with non-Indigenous students. This is despite fairly wide acceptance that the way teachers perceive students will impact on the teaching, learning and assessment outcomes that students receive (Wyatt-Smith, 1995). The research reported in this paper was conducted in remote Aboriginal communities throughout Queensland. It addresses how "white" teachers, who are usually young and newly graduated, view the mathematics learning of Aboriginal student, and how these perceptions differ for white students.

It is commonly accepted that education functions to reproduce social inequalities and that teachers have a role in this process (Bowles & Gintis, 1976). In Australia there is no greater case of social inequality than the gap between the Aboriginal and non-Aboriginal citizens of the country. In the 200 years since Invasion, Aboriginal culture has been systematically disavowed, disempowered and displaced. Despite often well meaning rhetoric, this degradation of Aboriginal populations has, in many instances, not improved since the Referendum of 1967. Government after government has successively failed to rectify the mistakes of the past, instead using welfare and monetary reparations as a way of attempting to silence Aboriginal voices. Recent High Court decisions, councils on reconciliation and more recent revelations about the current living conditions of many of Australia's Aboriginal peoples have gone some way to making governments and oppositions alike realise that Aboriginal voices will not be silenced. However, the policies and decisions of the past and those that continue have left many Aboriginal communities existing in a kind of limbo – existing neither within the traditional parameters of Aboriginal culture nor as part of the Western capitalist system which contextualises most of Australian society. This situation potentially provides one reason why Aboriginal communities are still plagued by issues of poverty, alcoholism, domestic violence, high mortality rates, high teenage pregnancies and a range of other social and health issues which ultimately threaten the survival of these communities (Queensland Government, 2001).

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Aboriginal Education in the Australian Context. Education has not only failed to level the playing field between black and white Australians, it has in fact widened the gap between the two groups. The issues in Aboriginal education may be seen as relating to the historical and continuing negative treatment of members of the Aboriginal community by Anglo-Australian society. In particular, the non-Aboriginal culture of the country has not recognised the various cultures of the Aboriginal communities. Education with its typically Eurocentric values has cemented this treatment by not acknowledging the knowledges that Aboriginal students bring to schooling and by further expecting 'black' Australians to jump through 'white' hoops in terms of achievement and assessment. Issues in education in Aboriginal communities are further complicated by growing health and social issues that leave many students unable to engage with school at the level required for success. Despite these obstacles, many Aboriginal students do negotiate the education system successfully. While this is due to many factors, it is no doubt helped by teachers who have taken the time to understand Aboriginal students and the way in which they learn. Developing effective teaching strategies that lead to effective outcomes for Aboriginal students has its origins in the ways in which the teachers perceive their students in terms of mathematics learning.

Teacher perceptions of students. While much has been written about teacher perceptions, surprisingly little concerns teacher perceptions of their students and the implications of these judgments, and even less deals with teacher perceptions of mathematics learning and Aboriginal students. The literature discusses how teacher perceptions of curriculum alter the way it comes to be implemented in classrooms (McLeskey & Waldon, 2002) and of the impact of teacher beliefs about the administration of a school on its successful organisation (LoVette & Watts, 2002). Writings about teacher perceptions of students occur largely in assessment theory where discussions tie perceptions to student achievement and, in particular, how knowledge about students become part of what Wyatt-Smith (1995) described as the "teacher knowledge files" utilised when teachers are making assessment judgments.

Some studies do deal with teacher perceptions generally, examining the various different perceptions that teachers have of their students. A study by Uhlenberg and Brown (2002), for example, looked at teacher perceptions as the reason for the black-white achievement gap in the United States. Of interest to this study is that while black teachers tended to lay the blame at the doorstep of schools and educational systems, white teachers tended to focus on students, parents and home environments. A further study by Drame (2002) collected data from 63 teachers to determine the extent to which socio-cultural variables affected teacher perceptions of classroom behaviour. The study concluded that teacher perceptions, particularly of learning disabilities and academic performances, affected teachers' instructional patterns and their dealing with students. A study conducted across four countries (Woolman, 2002) – India, Nigeria, United Kingdom and the United States – into dropout prevention discovered that reversing negative teacher perceptions about minority

children was one of the factors essential in keeping at-risk students in the education system. A further study (Mavropoulou & Padeliadu, 2002) found that rather than teacher perceptions of students being the primary determinant in affecting their behaviour in the classroom, it was in fact their own sense of behaviour and control that determined their treatment of students.

While few studies have been conducted into teacher perceptions of Australian Aboriginal students, Green (1982) did examine the influence of the classroom teacher on the performance of Aboriginal children. In this study, a group of 15 white teachers were asked to list the major differences between teaching Aboriginal and non-Aboriginal students. The teachers were drawn from classrooms where the Aboriginal school population varied from between 5-60%. The responses of the teachers were separated into five separate categories. All of the categories developed worked out of a deficit model whereby Aboriginal students were seen as having some form of 'insufficiency' that caused their low academic performance. Of the 80 responses coded, 48 of them felt that the students themselves were deficient - lacking interest and language skills, and not having proper behavioural skills, adequate nutrition and proper school socialisation skills. Eleven of the responses blamed familial difficulties including low parental expectations, little parental support and a transient lifestyle as being the cause of the school problems experienced by Aboriginal students. Six responses suggested that there was a cultural gap between Aboriginal and non-Aboriginal students and that this gap made it difficult for the students to assimilate into the schooling system, while four of the responses believed that the schools could be effective in teaching Aboriginal students if they received more assistance from government agencies. Few teachers saw the school as having any responsibility. As Green described,

> Eleven of the responses attributed the child's learning problems to factors for which the school could be deemed responsible and six of those were written by one teacher. The most common responses were 'Aboriginal children are ignored … reading materials are inappropriate … prejudice by teachers and non-Aboriginal children … teachers do not have special training to teach Aborigines … a lack of Aboriginal support staff … inadequate extra curricular activities'. (p. 111)

Generally, working as they were from perceived deficiency models, none of the teachers appeared to expect Aboriginal students to achieve at schooling. Similarly, not one of the teachers involved in the study suggested that, potentially, their own beliefs that students would fail was a contributing factor in that failure. There is no sense in the data presented that the teachers felt that they could make a difference to the way Aboriginal students learnt and the success that they experienced.

The view that teachers cannot improve the quality of Aboriginal education is directly contradicted by Malin's (1998) case study of a teacher who had an outstanding record of success teaching Aboriginal students. According to the parents of one of the Aboriginal students in her class:

> [Mrs Banks] had something interesting every day for the kids there and they really wanted to go to school [my son] talked a lot about school then, and he never wanted to come shopping with me because he was going something at school and he was really excited about it. It was the same when I asked him to stay home and look after the baby. He did it but he never wanted to. But now [he's not in Mrs Banks' class] he wants to stay home and look after the baby and he'd rather do that than go to school. (p. 245-246)

Malin attributed Mrs Banks' success, in part, to her innate knowledge of the students and her belief that students could learn if she supported them. Therefore her success along with Green's teachers' lack of success supports the hypothesis that teachers' perceptions of Aboriginal students and their learning – in particular seeing them positively and believing in their ability to learn - has a huge impact on the success that the teachers can achieve in a classroom.

The focus of this paper. This paper reports on the perceptions held by teachers in schools with 50-100% Aboriginal populations of Aboriginal and non-Aboriginal students as mathematics learners. The interviews from which the data were gathered this data were part of a larger project dealing with the mathematics professional development of teachers in remote North-Western Queensland schools.

METHOD

As part of the data collection to evaluate the effectiveness of the mathematics professional development program, 12 teachers at three schools in the remote North-West of Queensland were interviewed on two separate occasions, once early in the academic year and then again towards the end of the same year. The topics covered in the interview ranged from teachers' perceptions of mathematics teaching and learning to what mathematics curriculum content should be taught and what the teachers felt needed to be done in order to improve mathematics classroom teaching. It should be noted that all the teachers interviewed were white, young and inexperienced, having only graduated from university within the last four years.

At the second interview, teachers were asked to identify the mathematics learning differences between Aboriginal and non-Aboriginal students. This data was examined to identify any common threads that were apparent and whether or not there was any correlation between the responses of these teachers and the ones studied by Green twenty years ago.

RESULTS

The data indicated that some of Green's (1982) findings were still prevalent in modern teachers' talk about the differences between Aboriginal and non-Aboriginal students as mathematics learners, but there were some surprises.

Deficit perceptions. Six out of the twelve teachers' comments could still be classified as originating from a 'deficit' model particularly with respect to the students and/or their families. The two main deficits referred to by teachers were

school readiness and attendance. Teachers believed that Aboriginal students were not able to adapt to the culture of school because of what was described as a *lack of school readiness*. According to one teacher:

> *So you have to work so kids that come already knowing that you read, you listen to words and that you turn the pages, you know all that stuff that the kids do from 3. Where they can tell you, "I'm reading you the story!" They don't have any of that. And they haven't been read to and all that sort of stuff. So you've got to do so much work on school readiness! So that they're behind the 8 ball from the start. So that is, what I find is the biggest thing.*

As well, teachers saw lack of attendance and their transient lifestyles as a reason for Aboriginal students' mathematics and general learning difficulties and a major difference between Aboriginal and non-Aboriginal students.

The teachers attributed the lack of school readiness and attendance to a lack of interest in learning in the home environment. More than one teacher suggested that the real problem was that there was *no support for learning in the family* and that students were only sent to school to *get them out of their parents' hair* and that the only reason that the students themselves stayed at school was *to be with their friends*. According to one teacher:

> *I visited everyone of my kids' homes last year, I have never done that before, but I did it because I wanted to see where these kids were coming from. Eight or nine years of age, the home life that these kids are coming from, no wonder they would want to come to school. Just to get out of the house, no wonder they don't have that fostering of further education, they do home, ride their push bikes around, there is no place where they can just sit and talk because Mum and Dad are not interested or don't have the knowledge themselves.*

The teachers believed that Aboriginal students identified school not as a place to learn but as a place to socialise. This was seen as occurring because of deficiencies in both the students and the family.

Teachers referred to stories of Aboriginal students, for example, coming to school not only unable to read but without any knowledge of what a book is and that you had to turn the pages to support their perceptions. One teacher even stated that lack of interest in learning was a general characteristic of Aboriginal students except for one student who the teacher said had been brought up in a "European way". The teacher saw this upbringing as non Indigenous:

> *But as long as I have known him, he's had an amazing study habit and he's upbringing's been that, has been different to that of all Indigenous kids in this town. Very non Indigenous European upbringing in relation to Mum and Nana. Especially Nana's expectations. A very non Indigenous person herself. The way she speaks, the way she carries herself ...*

Teachers believed that Aboriginal students' lack of preparedness or knowledge of school meant that students were behind in comparison to their non-Aboriginal classmates who generally came with knowledge of 'school' and what happened there. There was a perception in some of the responses that the lack of readiness for school gave rise to behaviour problems such as the inability to sit still.

Non-deficit perceptions. While some of the comments above reflected similar attitudes to those uncovered by Green (1982), comments 6 teachers indicated that some progress had been made in understanding the way in which Aboriginal students learn.The comments that showed progress referred predominantly to the differences in mathematics learning style.

The most common perception was that Aboriginal students were *hands-on learners* in mathematics. Two teachers used the term *kinaesthetic* to describe the way in which Aboriginal students learn with one stating that *I have noticed with place value charts and things, making them touch, really made it sink in*. This teacher suggested that the tactile nature of the way in which students learnt mathematics was not given enough consideration in the design of lessons. Another teacher stated the same position but in a negative manner, stating that Aboriginal students found pen and paper work difficult and had low tolerance for board work and copying information.

One of the teachers interviewed believed that Aboriginal students tended to be more visual learners as a result of their hearing disabilities due to Otitis media, an inflammation in the middle ear that an estimated "30% and 80% of all school-aged children in remote Aboriginal communities" have (Queensland Government, 2001, p. 311). However, she also said that while auditory instructions were problematic, Aboriginal students *don't like writing they would rather discuss things*.

Other comments referred to lesson organisation. One of the teachers commented that *Aboriginal students learn best in structured learning environments* rather than undertaking work that required independent learning. Another teacher's comment suggested that Aboriginal students were not risk-takers with their learning and preferred guidance and support.

Interestingly, two teachers said that they found no real difference between Aboriginal and non-Aboriginal students in mathematics classrooms. One teacher, who taught mathematics to only four non-Aboriginal students out of a class of twenty-three, reported that all the students were there all the time regardless of their background, were equally keen and showed no *big gap* between the students. The implications of this teacher's comments seemed to be that absenteeism was a problem for some of the students and created a gap in achievement. Another teacher stated *I don't really think of them as being Aboriginal or non-Aboriginal. I just try to think of them as each person, their standard and how I can teach better each and every one of them at their level*. Finally, a third teacher felt that the students themselves weren't intrinsically different but they became different as result of the way they were treated. This teacher felt that educationally Aboriginal students became teacher-fulfilling

prophecies because white teachers didn't believe they had academic abilities. His perception was that low expectations in mathematics learning are producing low results but that if teachers raised the bar, Aboriginal students would rise to meet the challenge.

CONCLUSIONS AND DISCUSSION

Interestingly, when the teachers in this study were asked whether or not there were any mathematics learning differences between Aboriginal and non-Aboriginal students, many of the teachers spoke of achievement related issues such as attendance and school readiness rather than actual learning styles. Those who did speak of learning, however, stated that Aboriginal students appeared less interested in written work and more motivated by hands-on activities. They also suggested that Aboriginal students were visual learners and coped better with structured supported learning as opposed to individual, independent learning.

With the exception of Green (1982), it is difficult to view the findings of this presented here in the context of other literature as at this stage no correlation has been done in this study between teacher perceptions and their impact upon students' mathematics learning outcomes. In comparison to Green's research, while some teachers did identify with the deficit models that suggest that problems that Aboriginals have with education are a result of their own limitations and that of their families, many teachers now do seem to acknowledge that Aboriginal students have different styles of learning that should be recognised. There is also less blame accorded in this study either to the system or the environment than in Green, but similar to Green there was only one comment that indicates that the curriculum adopted by the school does not provide enough for Aboriginal learning styles.

Three distinct categories of responses have emerged in this analysis of teacher perceptions of Aboriginal learning styles. In the first category, teachers retain a stance that the problems of Aboriginal learning have causes that are deep-seated and cannot be solved by the school. In the second category, teachers identified that Aboriginals do have different learning styles. In the third category, teachers indicated that they saw no great difference between Aboriginal learning styles and those of non-Aboriginal students. How these perceptions come to affect teaching and learning and ultimately assessment outcomes is the subject of future analysis.

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TWO IMPORTANT INVARIANT TASKS IN SOLVING EQUATIONS: ANALIZING THE EQUATION AND CHECKING THE VALIDITY OF TRANSFORMATIONS.

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Our study concerns the analysis of teacher and student activities. Secondary school 6th grade students were confronted, for the firs time, with solving equations. We used our cognitive models of students and experts (in algebraic calculations) for analyzing the teaching process and the cognitive functioning of students. Our model led us to consider the management of mathematical justifications as a fundamental teachers' task. We believe that these models can become a daily tool for teachers, and principally for newer teachers

The analysis of teacher and student activities was identified as an important research object in PME conferences. Boaler, J.(2003) observed : « One interesting observation from our coding of class time was the increased time that teachers spent questioning the whole class in the reform classes. Whereas the teachers in the traditional classes gave students a lot of information; the teachers of the reform approach chose to draw information out of students, by presenting problems *and asking students questions*". A good teacher's activity can be resumed in the following manner: "when students were unsure how to proceed with open problems the teacher encouraged the students to engage in these practices: exploring, orienting, representing, generalizing and justifying. … *rather than deflecting her authority to the discipline : is this correct?"*

Teachers need to analyze problems and student cognitive functioning in order to be able to guide the construction of student competencies in mathematics. Teacher need cognitive models to understand and to anticipate student errors and difficulties. How can students be guided in constructing competencies? We attempt to answer to this question here.

This research is based on our previous researches, principally: "A cognitive model of experts' algebraic solving methods" (Cortés, A. (2003)) and "Solving equations and inequations, operational invariants and methods constructed by students" (Cortés, A. & Pfaff, N. (2000)). In these researches, we observed that most students (10th grade) use transformation rules without mathematical justification and that their solving methods resemble algorithms. In contrast, what makes teachers' solving methods effective is the mathematical justification of transformations. The respect of this fundamental characteristic guided the teaching process that we analyze in this article.

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The main goal of our study is to use our cognitive models in constructing a tool for analyzing student activities and the teaching process. To discuss the relevance or the best manner of teaching pre-algebra is not our goal.

The experimental class and the research. The official curriculum for sixth grade in France includes the solving of equations. We followed this prescription and our sixth grade students were confronted, for the first time, with solving equations of the type: x+a=b; x-a=b; x/a=b; a/x=b; ax+b=c; b-ax=c; where a, b and c are positive decimal numbers. Our students had never worked with numbers with sign (oriented numbers), so the solutions we asked them to calculate were never negative numbers. The sixth grade is the first year of the secondary cycle , the average age of our thirty students was eleven years old and students' level of knowledge was good. The experiment was conducted in a secondary school situated in the north-east of Paris. We experimented for three hours with solving of equations and the teacher of the class was Mrs. Kavafian. The data analyzed were the students' written work and the recorded interaction between the teacher and the students. During the experiment, students also solved word-problems that we do not analyze in this article.

THEORETICAL FRAMEWORK

The students explicitly explored the concept of the equation, the concept of the unknown and they used transformations for the first time. We focused our work on two tasks that we consider essential in algebraic calculations: the analysis of the mathematical object and the checking of the validity of transformations. In this theoretical framework we only included the used results of our previous researches and the theoretical concepts used in constructing the learning courses and in the analysis of data.

In Cortés A. (2003) five invariant tasks were identified. These are the tasks that the expert always carries out (implicitly or explicitly) in performing transformations. Each task is carried out by means of some specific piece of knowledge or by means of a competence, that we call the operational invariant of the task. The concept of operational invariant was introduced by Piaget (for example, Piaget (1950) considered the conservation principles in physics as operational invariants of physical thought). The invariant tasks of algebraic calculations, adapted to the solving of equations in our experiment are:

1- Analyzing the equation and choosing a transformation. The operational invariant is the concept of the equation.

The concept of the equation enables subjects to carry out the analyses which lead to choosing the right transformation. However, our students were being confronted with equalities in which the unknown was represented by a letter for the first time. So, during the experiment they explicitly began the construction of the concept of the equation. The teacher guided the students in conceptualizing the letter as an unknown number, the notation of multiplication as a juxtaposition of the letter and the

coefficient and, notably, the meaning of the equal sign: an equivalence which can either be "true" or "false".

The teacher guided the students in analyzing the equations appropriately according to the solving method. First, equations were solved by substituting numbers for the unknown. Later, equations were solved by means of transformations.

Frequently, the analysis of the equation lead the students to make inferences like "It's an addition, so I should subtract", "It's a multiplication, so I should divide", etc.; which are false in certain situations. These transformation rules are theorems in action (Vergnaud G. (1990)); they are mathematical properties that the students use automatically without mathematical justification.

2- Identifying the operation to be given priority.

The identification (usually implicit) of the operation to be given priority allows choosing a relevant transformation. There is a multiplicity of operations and the priority of operations depends on the situation. Subjects need to have knowledge concerning each pair of operations involved in a particular situation; in this sense the operational invariant is composite. Similar operational invariants were found by Pastré, P. (1997), in the area of cognitive ergonomics. In the present research, students were confronted with the priority of multiplication over addition and subtraction.

3- Checking the validity of the transformation. A mathematical justification of the transformation is the operational invariant of the task.

A mathematical justification of the transformation chosen establishes a link between the mathematical properties of the equation and the transformation. A mathematical justification allows subjects to check the validity of transformations. In this research the mathematical justifications used were, principally, the students' knowledge of arithmetic.

a- Operational invariants of the "principle of conservation" type: the conservation of the truth-values of equations.

At the beginning of the experiment equations were solved by substituting numbers for the unknown: the truth-values of the equation allowed students to identify the solution. This mathematical property is "self-justified" or "self-evident" for the students and is the mathematical justification of the procedure. Students implicitly used the conservation of the truth-value when they verified solutions obtained by transformations (by substitution in the given equation).

All authorized transformations conserve the truth-values of mathematical expressions and this property constitutes the more general mathematical justification and filiation of transformations in solving equations, inequations, systems of equations... We believe it is relevant to use this principle in teaching algebra from the very outset as well as in all types of algebraic calculation in secondary school.

b- *Operational invariants of the "self-evident mathematical property" type which students are able to justify*.

Mathematical justifications of additive transformations. Students summarized the addition of two decimal positive numbers in the following manner: "the addition of two little numbers makes a bigger number". Then, the biggest number was identified and, for students, this mathematical property was "self-evident". This property allowed students to analyze subtraction: the addition of two numbers, for example 67+22=89, makes two subtractions possible "the biggest minus a smaller number" (89-22=67; 89-67=22). Likewise, the subtraction of two numbers, for example 89-22=67, makes an addition possible, 67+22=89, whose truth-value allows checking the validity of the result of the subtraction. It is possible to check an addition by means of a subtraction but it is not usual.

Mathematical justifications of multiplicative transformations. The students were competent in multiplying and dividing integers. Their identification of the biggest number allowed transforming a multiplication into a division and vice-versa, for example, 7*8=56 leads to 56/7= 8 and 56/8=7. This property lead students to make errors with decimal numbers: 9.1=0.7x13 sometimes lead to 13/9.1=0.7. The main goal of our research was to explicitly introduce the tasks of analysis and the checking of the validity of transformations. To this end, we used division without any particular difficulties.

4- Checking transferred terms in a new written expression. This task is not relevant to solving the equations used in this research.

5- Numerical calculations. Most students could do numerical calculations without any particular difficulty. For other students, the solving of equations allowed them to reconstruct lost arithmetical knowledge.

The expert's algebraic solving methods. We summarize the main characteristics of expert's solving methods (Cortés A. (2003)). The analysis of the particularities of the mathematical object allows subjects to choose relevant transformations. For most exercises, teachers immediately choose a relevant transformation. They have reached a very high degree of expertise and they do not need to explicitly justify transformations: transformation rules are self-evident (they have the intimate conviction that they are true); it allows them to work quickly. But teachers are able to check the validity of transformations explicitly: they know the mathematical justifications of the transformations they use.

Some situations are not self-evident for teachers and when they are confronted with the choice of one transformation among several, they explicitly check the validity of transformations by means a "self-evident " mathematical property. This aspect of the experts' functioning constitutes a relevant model for the analysis of learning and teaching processes.

EXPERIMENTAL WORK

In solving equations in which the unknown was represented by a letter, students explicitly explored the concept of equation and used two solving methods.

EXERCISE 1 - Find the numerical value of the number n: 56/n=7; n= ; 22-n=11, n= ; n+8=18, n= ; 108/n=12, n= ; 25/n=5 n= ; 200-2*n=88, n= ; n*1.2=12, n= ; 50-n=46, n= ; 3*n+5=23, n= ; n*42=0, n= .

The analysis of the equation: the teacher asked the class to read the equations together before doing any calculations. The first equation was read as a question, this allowed analyzing the meaning of the letter. Students read the equation as an unknown result, for example "how much is n?"; "which result is n?". One student managed to express the question as "what number represents the unknown, n".

The analysis and the solving of the equation. The first equation was read as a numerical operation: a student said "56 divided by n makes 7". Students were given the solution "n is equal to 8" and were asked to check the validity of the result by the truth-value of the equation (56/8=7). The teacher summarized the reading of the second equation as "22 minus a number makes 11". Students substituted numbers for the unknown (22-11=11) and answered "n is equal to 11". For easy equations, most students proceeded by substituting numerical values until they could find the value that satisfied the truth value of the equation, the truth value being what allowed them to identify the solution.

Solving the equations through transformations. Analyzing equations lead students to make inferences: "It's an addition, so I should subtract", "It's a multiplication, so I should divide", etc. These rules, at that time, were theorems in action. For example, some students mentally transformed the equation 56/n=7 into 56=n*7 (they did not write the new equation), and thus used multiplication for solving the equation. Similarly, for the equation 22-n=11 which was transformed into 22= n+11, the solution was calculated by addition. In the solving of the equation 108/n=12, some students suggested "the solution is 9; 108 must be divided by 12".

Identification of the operation to be given priority. Transformation rules lead to errors. For example, in order to solve 200-2n=88, two students proposed to the class: "200-2=198 and 198-88=110". The teacher analyzed the equation taking into account the priority of the multiplication: "200 minus two times something(2n) is equal to 110"; "give numerical values to n and see what you can find". The students made the correct calculation: "200 minus something is 88... so it's 112...and 112/2 is 56". The teacher asked them to check the validity of this solution: "Are you sure, what do you have to do to be sure?". The students suggested that "200-(2*56) must be equal to 88". The students implicitly used the conservation of the truth-value as justification.

Exercise 2: The solution and the truth-value of the equation: only one solution is possible.

The following equations: a) 3n+15=27; b) 78-5x=46; c) 42-3n=2n+27, were implicitly analyzed as the equality of two functions. Students completed tables in which several values were given to the unknown. The two members of the equation were compared: the equality was either "true" or "false".

Students explored the variation of the two members of the equations and were able to determine and justify the fact that there was only one solution (we used equations which have a single solution). For de first equation the justification was "…it keeps going up and will never be 27". For the second equation the justification was "When x is bigger than 24.8, it keeps going down and never will be 46". For the third equation the justification was more elaborate: "One side of the equation goes up and the other goes down, they meet only once". Students wrote these justifications down.

Exercises 3 and 4 - Using transformations and checking the validity of the transformations.

Equations were solved by writing transformations, for example: 67+n=125; n=125-67; n=58. The written transformations allowed the teacher to perform the checking of the written numerical operations using a student's arithmetical justifications. The teacher established a strong tutorial activity: she asked the class to write the transformations down before performing the numerical calculations. In exercise 3, students solved the following equations: 33+n=150; 4.5+n=9.2; n-22=67; x-33=99; 5y=135; 10m=66; n/11=33; z/6=56. In exercise 4: 67+n=125; 220+2n=9000; 4z-16=8; 200-n=88; 6.6y=132; 250=5t-35; n/3.1=7; n/6+20=90.

Checking the validity of additive transformations. In solving the first equation 33+n=150, the teacher analyzed the equation in a particular manner: "n is the number which we must add to 33 to obtain 150, is this the definition of a subtraction?" The students answered, "n is the difference between 150 and 33". The equation n-22=67 was analyzed as a subtraction, which can be verified by an addition: 67+22=n. *The analysis of the equation as a numerical operation, which can be transformed into another operation, allows choosing a relevant transformation* (notably, the numerical calculation of the solution). At the same time, arithmetical justification allows checking the validity of the transformation.

Certain students applied transformation rules which lead to errors. For example, the equation 200-n=88 was transformed into n=200+88. The teacher guided the students in constructing another equation, an addition (the verification of the subtraction) "200 is equal to 88+n". In order to solve the equation 4z-16=8, one must first calculate the number 4z. The teacher guided students' analysis, which ended in "4z is the biggest number. If I subtract 16, the result is 8; so 4z is equal to 8+16" ...."z is equal to 24/4".

Checking the validity of multiplicative transformations. For solving equations such as 5y=135, students used rules like “It’s a multiplication, so I should divide”. They proposed that “y is equal to 135 divided by 5”. They did not check the transformation, they checked the solution by verifying the truth-value of the equation:

"Once y has been calculated, you can multiply 5 by y and should get 135)". The teacher analyzed the equation appropriately, this allowed checking the transformation: "In the equation 5y=135, we have the numerical value of 5 times y and we need to know the value of only one y, so y=135/5". The equation was considered as a particular case of proportionality. To solve the equation n/11=33, students proposed, "n=33x11". When the teacher asked for a more analytic reading of the equation, students came up with "If you divide n into eleven equal parts, the value of each part is 33"; "you multiply n by the number of parts*". The teacher's tutorial activity obliged students to perform only those transformations that they could justify.*

A month later, the students were given a post-test. We observed seven errors in the solving of 3,7-y=0,1 (y=0,1+3,7) and x/2+5=8 (x=4/2). Students made small number of errors in the solving of the ten other equations (similar to Exercise 4).

CONCLUSION. In research and in constructing teaching courses, it is fundamental to analyze both the problems and the subject's solving activity. Identifying the invariant tasks constitutes a tool for the analysis of a subject's solving methods.

An operational invariant was associated to each invariant task, this is the mathematical knowledge that allows subjects to perform the task. This definition of the theoretical concept of the operational invariant enables us to explore and to understand the nature of operational mathematical knowledge.

We focussed our research on the analysis and the justification tasks, which are essential to solving mathematical problems. The students' arithmetical knowledge was used to justify the transformations performed. In order to solve equations, the students analyzed numerical operations in different manners, thus constituting another approach to arithmetic which enabled most of them to develop their mathematical knowledge, and, enabled still other students to reconstruct or to learn some of those arithmetic properties. In general, in teaching new concepts, it is possible to develop or to reconstruct students' previous knowledge.

We consider that students must be made to realize the advantage of working with mathematical properties that they are able to justify through daily work in class. The transformation of student theorems in action into justified mathematical knowledge is the way to build operational thought. This was the main goal of this research, because algebraic calculations will be presented differently later. Indeed, solving equations, inequations and systems of equations that include other types of numbers (numbers with sign, fractions…) is made possible by means of more general transformation rules and justifications. But the tasks of analyzing and checking the transformations remain invariant.

The teacher of the class had much experience and our research did not radically change her work. But our theoretical framework allowed her, notably, to refine the questions which guided students in conceptualizing new procedures.

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UNCERTAINTY DURING THE EARLY STAGES OF PROBLEM SOLVING

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This paper discusses the role of uncertainty during the early stages of problem solving. It is argued that students start the problem solving activity with some degree of uncertainty that may vary from high to low. This degree of uncertainty may affect students' decisions at early stages of the problem solving process. It may be suggested that an awareness of the possible effects of uncertainty may better prepare students to approach problem solving and support them to start building their own problem solving strategies.

INTRODUCTION

The idea of looking at uncertainty during problem solving emerged from an attempt at conceptualising what students do as they tackle non-routine mathematical problems. By analysing students' work it was observed that, at the outset, students' problem solving processes took place in a context of uncertainty. In other words, it was observed that problem solving usually started as a situation in which actions had to be taken in spite of insufficient knowledge. It was also observed that uncertainty was present in different degrees and that this led to different consequences.

A general look into the study of dealing with uncertain situations suggests that its importance has long been recognized in other areas. In business management, for instance, the study of uncertainty can be traced back to at least the late 1950's. In this area, the focus on uncertainty has evolved from trying to find ways of reducing it, to tapping into it as a source of creativity and stimulation (Jauch & Kraft, 1986; Ogilvie, 1998). Due to its role in modern organisations, attempts at modelling uncertainty (e.g., Downey & Slocum, 1975) have been, and may continue to be, conducted.

Uncertainty in mathematical situations has been considered in various studies. The most common position that has been adopted towards the study of uncertainty in mathematics education has been to acknowledge it (e.g., Rickard, 1996) or to explore ways of making positive use of it in didactic situations (e.g., Hadas & Hershkowitz, 1999; Hadas, Hershkowitz, & Schwarz, 2000). Few studies have attempted to develop models of uncertainty and to explain its relation to other aspects of learning and doing mathematics. Studies of this sort may provide useful information for those studies in which uncertainty emerges as a result or plays an important role.

This aim of this report is to discuss uncertainty as observed in a study of mathematical problem solving. The model of uncertainty that will be presented here is part of the broader model that emerged from this study. This sub-model aims at

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conceptualising the main aspects of uncertainty and the way in which they relate and affect each other.

THE CONTEXT: A BROADER STUDY

The results presented here are part of a broader study which investigated students' problem solving processes. The aim of this study was to develop a model that explains how students deal with non-routine mathematical problems. The study made use of the grounded theory methodology (Glaser & Strauss, 1967) and, particularly, of the methods of data analysis described in Glaser (1978; 1998). In general, grounded theory can be summarised as a methodology that consists of constantly comparing data (see Glaser, 1992). The method of constant comparison allows the researcher to generate categories and to start hypothesising about the way in which they are related. Emergent hypotheses are then compared against further data and thus a theory is developed. Grounded theory is, thus, an inductive methodology in which general patterns are derived from, and grounded in, the data.

This study was conducted within the context of a problem-solving course offered at the University of Warwick. The students that participated in the course were undergraduate students doing mathematics, computer sciences or (four-year) teacher training degrees. The aim of the course was to allow students to reflect on their "mathematical thinking and to identify and develop their own problem-solving strategies".

During the course, students were required to work on a number of non-routine mathematical problems and to document their thinking processes as they occurred. This provided students with a written document of their work and made it possible for them to review and share their ideas with the rest of the group. From the researcher's perspective, this provided rich data about the way students solve problems as well as some insights into their beliefs and abilities in relation to mathematics. Students' written accounts, or 'rubrics' (see Mason, Burton, & Stacey, 1982), were used as the main source of data for this study. Observations were also made during each session and informal interviews were conducted with a number of students.

The problems that students had to tackle may be described as non-routine mathematical problems, i.e., problems for which students knew that they were not expected to use any procedure or knowledge in particular. Also, some problems can be considered 'open ended' in the sense that students had decide what was it exactly that they wanted to achieve as a solution (see appendix 1)[i].

GENERAL RESULTS: A FRAMEWORK FOR LOOKING AT UNCERTAINTY

The analysis of students' rubrics during the main study gave rise to a considerable number of concepts about what students do with during problem solving. Finding suitable terms for these emerging concepts was, in many cases, a difficult task.

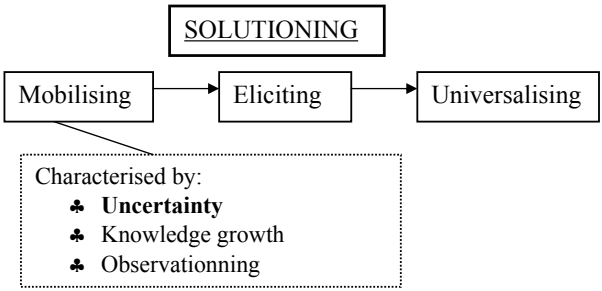
Furthermore, in the cases where terms were readily available, variations in basic aspects of their definitions made it difficult to make use of them. For this reason, the reader will notice that the study introduces of a number terms that, at first sight, may appear strange or even unappealing. These terms were chosen because, at the time, they were the most suitable way of illustrating and naming the concepts that emerged. They will continue to be in use until more suitable ones are found or until new data suggests that they need to be modified.

The following is a brief outline of the general results of the study. Due limitations in space, the concepts that will be introduced cannot be fully discussed. The outline, however, must serve to give a general picture of the framework from which uncertainty is being considered.

A model of students problem solving processes

An analysis of students' rubrics suggested that problem solving is best conceptualised as a cognitive process. This process was called 'solutioning' to highlight the on going nature of the situation. Solutioning consists of three main stages, namely, *mobilising*, *eliciting* and *universalising*. Mobilising is the stage at which information and understanding start to be generated in order to start solving the problem. Eliciting refers to the stage at which students go beyond the data that is given and what is being observed and start developing a solution. Universalising is the last stage whereby elicited solutions are improved and transformed into more general, rigorous and compact results.

For most students, the aim at the initial stages of problem solving is, in general, to generate data, information and understanding. Thus, it may be said that solutioning starts with mobilising. As said, mobilising is the process through which students start dealing with the situation, learning about it and understanding it. Mobilising is characterised by *uncertainty* and has two main consequences, namely, *knowledge growth* and *observationning*. The following figure provides a general representation of the model described above.



UNCERTAINTY

Students start to 'solution' by mobilising. Mobilising may start to be conducted in a context of uncertainty. *A context of uncertainty is one in which students lack the knowledge and understanding necessary to know exactly what to do next in order to deal with the situation.* In spite of this lack of knowledge, students are required to make decisions and take actions if they want to eventually provide a solution.

Uncertainty can be analysed from students' perspectives as they experience it. Students express uncertainty by raising questions about the situation or by making decisions based on insufficient or inadequate information. In the following example, Hannah started by asking herself some questions about the situation and trying to provide answers to them. In the case of the first question, her queries were easily resolved. As for the second question, the answer that she provided was based on the intuitive information she had at her disposal.

> What is the question asking me? I want a rectangular piece of paper, and take away from it the largest possible square. I then want to repeat the process with the left-over rectangle. I want to know what different things can happen, and when they will.
>
> Is there anything else I want to know? Well, I want to know where to stop–do I only repeat the process once or do I keep going until I reach and endpoint. I think I'll only repeat it once or otherwise the process could be infinite! So, I want to know what the different things are that could happen. (Hannah, Square Take Away, p. 1)

Uncertainty can vary in degree from high to low (and in some cases – where a student believes they know a precise, direct route to a solution – it can be null). Students may meet the problem with little or no knowledge about the situation, i.e., with a high degree of uncertainty. On the other hand, they may be presented with what they perceive as a familiar problem or with a problem that refers to material with which they are already experienced. In this case, there is a lower degree uncertainty. A lower degree of uncertainty means that the student has some ideas about the situation and about how to start tackling it.

The following quotes briefly illustrate cases of high and low uncertainty. The first example corresponds to a case of high uncertainty (as expressed by the students) whereas the last two correspond to cases in which uncertainty seemed to be low.

> Having established what exactly the question is asking me, I feel this problem is going to be incredibly difficult. I could literally place 2002 pieces of paper in the hat, however, there are some many possibilities of what number can be drawn out that I don't know how this would help. (Gina, Hat Numbers, p. 1)
>
> I can immediately see that this problem is very simple if I use squares, which are just regular rectangles. So I will first look at the squares briefly… (Jasmine, Diagonals of a Rectangle, p. 1)
>
> From first looking at the problem I can see that the answer will have something to do with common factors. (Jasmine, Visible Points, p. 8)

The degree of uncertainty is related to the type of decisions that students make during mobilising. Mobilising requires students to make decisions about what ideas to explore, about what to focus on. These decisions may vary from arbitrary to well informed. Arbitrary decisions are decisions for which there is insufficient information to make a choice and where students end up deciding on the basis of less relevant factors (e.g., choosing a course of action over another because it seems easier). High uncertainty means that decisions will be made arbitrarily, almost at random. As uncertainty decreases, however, decisions are likely to become more informed.

From the two examples below, the first one belongs to a student that seemed to be considerably uncertain about the situation. His decision on what to focus on at that time seemed to be more arbitrary than well informed. As for the second example, it belongs to a situation of low uncertainty. As it can be observed, the decision that the student made about how to proceed seemed a more informed one.

> Well, the question of the problem is perhaps ambiguous. I could make out of this question that I will aim to find what sizes work or what particular configurations work. For now I will concentrate more on the sizes (dimension) problem. (Marcus, Faulty Rectangles, p. 1)

> We know (a+b), (b+c), (c+a), but we want to know a, b, c explicitly. A solution to this problem would be to express a, b, c in terms of (a+b), (b+c), (c+a). (Alan, Arithmagons, p. 1)

The decisions that students make during mobilising may or may not lead to sustainable courses of action. A sustainable course of action or idea is one that turns out to be manageable for the student and that leads to generating results. High uncertainty means that students will be unsure about whether a particular course of action will be sustainable or not; they will need to engage in it in order to find out. Furthermore, the less uncertain the situation is, the more likely it is that the student will choose a sustainable course of action. The consequence of choosing a course of action that is not sustainable is that it will be eventually abandoned and that a new one will need to be found.

The following quotation illustrates how a decision taken in a context of uncertainty led to an unsustainable course of action. Realising that the chosen course of action was not going to be very productive provided the student with some experience that might have helped towards decreasing her uncertainty. Nonetheless, the student had to abandon the present course of action and face the need of finding other possible avenues to pursue.

> I will start by writing a grid of numbers, and using it to specialise systematically to find sums of lines with different gradients and different starting points...

> Stuck! I don't think this will work (i.e., I don't think I will be able to find any patterns) using this method, because the amount of numbers in the sum does not necessarily increase as you move down the table (i.e., as you start with a higher number) because the line does not necessarily pass through a number in the top row. (Hillary, Sums of Diagonals, pp. 1–2)

Arbitrary decisions made in contexts of uncertainty are not as valued as informed decisions based on logical deductions. However, making an arbitrary decision may be the best option for a student dealing with a highly uncertain situation. On the one hand arbitrary decisions may lead to an unsustainable course of action. On the other, arbitrary decisions may also bring unexpected knowledge and understanding and allow the student to open promising avenues. This suggests that, in some situations, making arbitrary decisions can be a good action after all.

Reducing complexity[ii] is an important aspect of mobilising that seems to help students deal with uncertainty. By reducing complexity, students focus on specific and relatively simple aspects of the situation. In students' words, they "start at the beginning", looking at simple cases or examples first. Students' aim in reducing complexity is to gather information and to gain understanding in order to be able to eventually move on to more sophisticated cases. Reducing complexity helps students deal with uncertainty by allowing them to focus on manageable aspects of the situation and helping them to start gaining knowledge and understanding.

In the example quoted below, the student was relatively uncertain about the situation. Reducing complexity helped her to deal with some of this uncertainty by allowing her to start learning about the situation.

> As m, n increase, what percentage of points is visible from (0, 0)?
>
> Stuck! I have no idea what the percentage should be as it would vary when m and n are varied.
>
> Let me try to draw some planes with different sizes…
>
> From the first trial, I can see that the percentage that we want to know is:
>
> (visible points on a plane/visible+invisible points on a plane)x100. (Karina, Visible Points, p. 1)

Another important aspect of uncertainty is the way if affects *observationning*. As a result of mobilising, students start noticing salient facts and pointing them out as observations. Observationning is about noticing and making a note of these facts (thus the term *observation-ing*). As a result of uncertainty, observationning, especially at early stages, can be exhaustive, meaning that students will try to make a note most of what is being observed. In other words, students may engage in noting most (if not all) seemingly salient facts or ideas "just in case" they are useful or relevant at a later time. The less the student knows about the situation, i.e., the higher the degree of uncertainty, the more likely it is that observationning will be conducted in this way.

The following example illustrates how uncertainty affects observationning. It may be suggested that it was due to uncertainty that, at the start of her process, Carolyn chose to reflect on the first observation made. It can be speculated that, not knowing too much about the situation led her to closely consider a variable that would later be considered irrelevant.

> I have noticed that there is a line of symmetry running through the grid from the top left through to the bottom right [see appendix]. Does this mean that, for example, 9 to 5 will give the same result as 5 to 9? Will diagonals that go to and from the same number (e.g., 4 to 4) need a different formula than those that go to a different number (e.g., 9 to 5)? … Hopefully I will be able to answer these questions by the end of this investigation. (Carolyn, Sums of Diagonals, p. 1)

Finally, uncertainty decreases as the student's knowledge and understanding of the situation increase. In other words, uncertainty decreases with *knowledge growth*. Knowledge growth is the change in what students know about the situation and in the way they deal with it. Knowledge growth is usually set off as a consequence of mobilising, however, it is not limited to it. Knowledge growth can also be the result of any other action that brings understanding and generates information. Since knowledge growth increases what students know and improves their understanding of the situation, it may help to reduce uncertainty.

IMPLICATIONS OF THE STUDY

The model of uncertainty proposed in this report suggests that the more uncertain the situation experienced by students during problem solving, the more likely it is that they will have to make arbitrary decisions. Moreover, it suggests that arbitrary decisions are less likely to lead to sustainable courses of actions than informed decisions. The model also suggests that students' uncertainty decreases with knowledge growth. This knowledge growth is a consequence of mobilising but it may also be said that uncertainty can serve to stimulate it. For instance, the exhaustive way in which students conduct observationing at early stages seems to be the result of working in an uncertain environment.

Recognising the fact that uncertainty may be present in various degrees can help students prepare for best dealing with it. By this, it is neither suggested that a context of uncertainty is a negative situation nor that it should be avoided. In fact, it seems that that students could benefit from learning to tolerate some uncertainty and even from actively creating certain levels of it. As suggested by other areas, a context of uncertainty can be a stimulating environment in which interesting questions can be raised and novel perspectives can emerge (see, e.g., Schoemaker, 2002).

Initial uncertainty may be a characteristic not only of problem solving but of the mathematical activity in general. Further studies can help to explain the role of uncertainty in other areas of mathematics as well. Such studies would help teachers and students to better understand the role of uncertainty and would support them in developing strategies for effectively tapping into uncertain situation.

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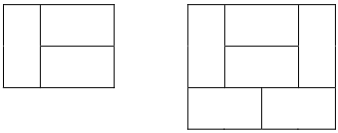
[i] The following are some examples of the problems students tackled during the problem solving course:

Square Take Away: Take a rectangular piece of paper and remove from it the largest possible square. Repeat the process with the left-over rectangle. What different things can happen? Can you predict when they will happen?

Hat Numbers: A hat contains 1992 pieces of paper numbered 1 through 1992. A person draws two pieces of paper at random from the hat. The smaller of the two numbers drawn is subtracted from the larger. That difference is written on a new piece of paper which is placed in the hat. The process is repeated until one piece of paper remains. What can you tell about the last piece of paper left?

Faulty Rectangles: These rectangles are made from 'dominoes' (2 by 1 rectangles). Each of these large rectangles has a 'fault line' (a straight line joining opposite sides).



What fault free rectangles can be made?

[ii] *Reducing complexity* is a term that seems to increasingly point towards *specializing* (see Mason, Burton, & Stacey, 1982). For the purposes of this study, the former term was considered more appropriate to convey students' main concerns as they focus on specific or simplified aspects of the situation.

STUDYING THE MATHEMATICAL CONCEPT OF IMPLICATION THROUGH A PROBLEM ON WRITTEN PROOFS

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In this paper, we present a didactic analysis of the mathematical concept of implication under three points of view : sets, formal logic, deductive reasoning. For this study, our hypothesis is that most of the difficulties and mistakes, as well in the use of implication as in its understanding, are due to the lack of links in education between those three points of view. This article is in the continuation of one previously published in the acts of PME 26. We present here the analysis of another problem from our experimentation. We want to show how a work on written proofs can allow a work on implication. Then we conclude with some transcripts.

INTRODUCTION

The existence of the implication as an object of natural logic, leads to confuse it with the mathematical object. As a result, the implication seems to be a clear object. Yet, students have difficulties related to this concept until the end of university, especially with regard to necessary conditions and sufficient conditions. Moreover, though it is in the heart of any mathematical activity, it is hardly ever taught in French teaching.

Our theoretical framework is placed in the theory of french didactics, in particular, we use the tools of Vergnaud's conceptuals fields theory and those of Brousseau's didactical situations theory. Our study is based on the work of V. Durand-Guerrier [Durand-Guerrier, 1999] on the one hand and of J. Rolland [Rolland, 1998] on the other hand. V. Durand-Guerrier shows, in particular, the importance of the contingent statements for the comprehension of the implication. J. Rolland, as for him, was interested in the distinction between sufficient condition and necessary condition.

This study is a part of our thesis on the mathematical concept of implication. It follows and supplements the study presented at PME 26. We present three points of view on the implication, a mathematical and didactical analysis of a problem on written proofs and conclude with some transcripts.

THREE POINTS OF VIEW ON THE IMPLICATION

This paragraph was detailed in our previous research report in PME 26. Yet, we think this part of our research is necessary for the reader to understand the following problem and the aim of our research hypothesis.

The mathematical implication seems to be a model of the natural logic implication we use in our everyday life. Like any model, this mathematical concept is faithful from certain angles to that of natural logic but not from others. This distance between the mathematical concept and the natural one leads to obstacles in the use of the

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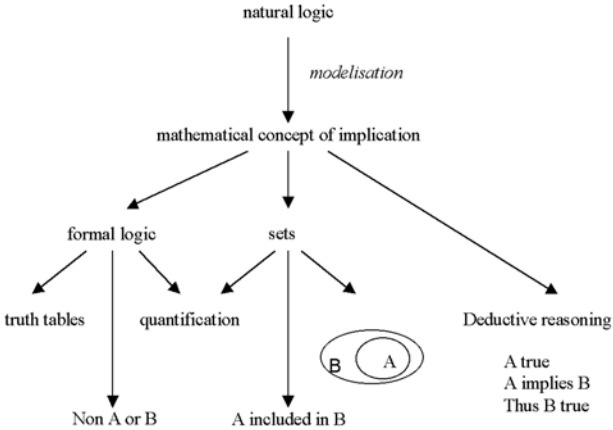
mathematical concept. An epistemological analysis [Deloustal, 2000] enabled us to distinguish three points of view on the implication : formal logic point of view, deductive reasoning point of view, sets point of view.



Of course, these three points of view are linked and their intersections are not empty. We will not develop here the formal logic point of view (for example truth tables or formal writing of the implication).

We call "deductive reasoning" the structure of an inference step : "A is true ; A implies B is true ; Thus B is true". Its ternary structure includes a premise "A is true", the reference to an established knowledge "A $\Rightarrow$ B" and a conclusion "B is true" [Duval, 1993, p 44]. The reference statement may be a theorem, a property, a definition, etc. One thus builds a chain of inference steps : the proposition obtained as the conclusion of a given step is "recycled" as the entrance proposition of the following step. Therefore, in the deductive reasoning, the implication object is used only as a tool. However, in French secondary education, where this point of view is the only one, it often acts as a definition for the implication.

Generally speaking, having a sets point of view, means to consider that properties define sets of objects : to each property corresponds a set, the set of the objects which satisfy this property. The sets point of view on the implication can then be expressed as follows : in the set ***E***, if ***A*** and ***B*** are respectively the set of objects satisfying the property A and the set of objects satisfying the property B. Then, the implication of B by A (i.e. A $\Rightarrow$ B) is satisfied by all the objects of the set ***E*** excluded those which are in ***A*** without being in ***B***, i.e. by all the objects located in the area shaded here after.

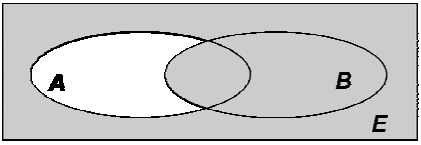


Figure 1

RESEARCH HYPOTHESIS

The experiments carried out for three years, within the framework of our research, have shown that the implication was not a clear object even for beginner teachers. Moreover, they showed that, contrary to a widespread idea, a logic lecture is not enough to get rid of these mistakes and difficulties.

Following these comments, we formulate the research hypothesis : it is necessary to know and establish links between these three points of view on the implication for a good apprehension and a correct use of it.

In the following paragraph we show that a problem on written proofs, using only easy properties, may question the reasoning in a non obvious way.

CONDITIONS OF THE EXPERIMENTATION

The problem we present results from an experimentation carried out in 2001 with beginner teachers of mathematics. We worked with two groups of approximately 25 students at the IUFM[1] of Grenoble and Chambéry (France). This experimentation includes two three-hour-sessions on the proof and, in particular, on the implication. The first session contained two problems (one in geometry, one on pavings), the second one proposed a work on written proofs. For each meeting, a work by groups of three or four people was following an individual work to allow questionings and discussions. We presented, in PME 26, a problem of geometry resulting from the first session. We present, now, a problem on proofs, following the previous one.

Before beginning the analysis of this new problem, we want to remind the reader of the problem the beginner teachers had to solve in the previous session :

Let ABCD be a quadrilateral with two opposite sides having the same length. What conditions must diagonals satisfy to have : (P3) two same-lengthed other sides ?[2]

PRESENTATION OF THE PROBLEM

Here is the new problem as we gave it to the students:

Let us remind the previous problem :

Let ABCD be a quadrilateral with two opposite sides having the same length. What conditions must diagonals satisfy to have : two same-lengthed other sides[3] (P3) ?

A necessary and sufficient condition is : “the diagonals cut in their middle” (We call it C1)

What do you think about the following dialogue ?

X : Look, the condition “one of the diagonals cuts the other in its middle” is maybe also a necessary and sufficient condition ? (We call it C2)

Y : Impossible, since this condition C2 is strictly weaker[4] than the other (C1).

MATHEMATICAL ANALYSIS OF THE TASK

Let us call *Q*: the set of quadrilaterals ; *H*: the set of quadrilaterals with two same-lengthed sides (H: the property of having two same-lengthed sides) ; *NC*: the set of non crossed quadrilaterals ; *Cvx*: the set of convex quadrilaterals.

Discussion between X and Y

The argument of Y is not a valid argument. Indeed, two conditions can be equivalent on a subset even if they are not usually equivalent. For example, in the set of parallelograms, the condition “having one 90 degrees angle” is equivalent to the conditions “having four 90 degrees angles”.

In this problem, C1 could be equivalent to C2, to deny it one must prove it is false on the mathematical objects. Under a logical point of view Y is wrong.

Implications between C1 and C2

We can translate the first question (from X) by: “ In the subset *H*, is the equivalence C1⇔ C2 true ? ” or “ Is the equivalence [C1 and H] ⇔ [C2 and H] true ? ”

In our problem, there is no equivalence between C1 and C2. We present two counter-examples, i.e. two quadrilaterals which satisfy H and C2 but not C1, figures 1 and 2.

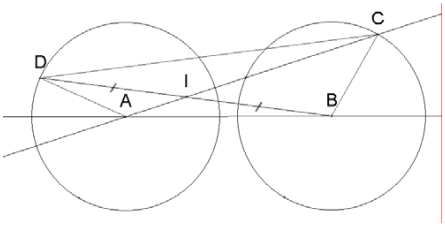


Figure 2

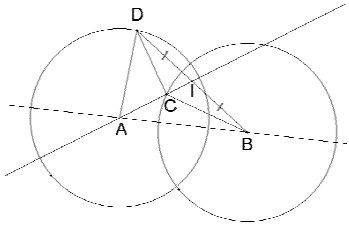


Figure 3

In this paper, we have no time to describe the set in which the two conditions are equivalent. But, we can notice that it is an interesting question for the beginner teachers who want to solve the problem.

Set of objects satisfying H and P3

To allow the reader to tackle the problem with the same knowledges as the beginner teachers, we give the solution of the previous problem. We do not detail it for lack of time but the reader can convince himself easily.

The objects satisfying both H and P3 have two same-lengthed sides and two other same-lengted sides, we have then the equivalence:

H and P3 $\Leftrightarrow$ Parallelogram OR Crossed Quadrilateral called CQ [figure 4]

Figure 4

Implications between C1 and P3

In our problem, the assertion "A necessary and sufficient condition is that the diagonals cut in their middle " is false. Indeed, in the set *H*, C1 is not a necessary condition to P3, since C1 is equivalent to the condition "to be a paralellogramm".

$$\text{H and P3} \Leftrightarrow \begin{cases} \text{Parallelogram} & \Leftrightarrow \text{C1 (diagonals cut in their middle)} \\ \text{or} & \\ \text{Crossed Quadrilateral (CQ)} & \end{cases}$$

That is to say that, in *H*, C1 implies P3 (C1 is sufficient for P3), but P3 does not imply C1 (C1 is not necessary for P3) as shows the counter-example (CQ).

On the other hand, if we place the problem in the set *NC* (non crossed quadrilaterals), there are then the equivalences : H and P3 $\Leftrightarrow$ Parallelogram $\Leftrightarrow$ C1. That is to say, in the set $H \cap NC$, C1 is a necessary and sufficient condition for P3.

Implications between C2 and P3

According to what we said previously, in $H \cap NC$, there are the implications:

H and P3 $\Leftrightarrow$ C1 $\Rightarrow$ C2, that is to say that C2 is then necessary to P3 but not sufficient as shown by the counter-examples figures 2 and 3.

DIDACTICAL ANALYSIS OF THE SITUATION

We present now the choices we made for this problem in terms of didactical variables. A didactical variable (DV) is a characteristic of a problem likely to involve, according to the values alloted to it, various strategies of resolution by students. All the variables of a problem are not didactical variables, the various strategies involved must be really different compared to the aimed learning. We want to show the variables wich allow a work on implication and especially those linked to the three points of view.

General choices

DV1: Mathematical framework for the problem

First of all, we choose, for our experimentations, very easily accessible mathematical concepts. Indeed, our hypothesis is that to see a work on the reasoning there must not be difficulties linked to a mathematical concept, to be able to distinguish difficulties due to the concept of implication. In this problem, there are only mathematical notions well known by students such as quadrilaterals, parallelograms, diagonals...

Moreover, we chose to place this problem within a geometrical framework. Another problem of our experimentation concerns pavings, it is easily accessible and is appropriate as for previous requirements. But we wanted to show them, since they are teachers, that even with a taught concept, like geometry, pupils can study the reasoning in a non usual way.

DV2: Practical organization of the session

Our hypothesis is that a research in groups is necessary for our problem. That allows a confrontation between the various points of view, in particular logical and deductive reasoning points of view. Furthermore, it stimulates discussions. Nevertheless, the first individual work gives to each one time to have his own idea about the problem. These various ideas will feed discussions.

Choices for this situation

DV3: point of view on the implication

Our hypothesis is that, to allow discussions, the problem must be in a mathematical context and not only in a logical one. Indeed, our previous experimentations with university students have shown that logical knowledges can coexist with false conceptions on implication. That is to say that, in case of a problem in an only logical context, all students could agree on the right solution without showing any difficulty. Therefore, we chose to place our question in a geometry context, in order to confront the formal logic point of view and the deductive reasoning point of view. However, since beginner teachers have studied the problem during the last session, we do the hypothesis that the difficulties of geometry should not be an obstacle to the discussion.

We want to know if, within the framework of a proof in geometry, the students are able to work only under one logical point of view without using mathematical properties. In response to our questioning, we will thus distinguish three types of answers:

The first one based on logical point of view: Y is wrong, this can be possible.

The second one linked to the mathematical contents: Y is right, in the subset *H* of this problem, C2 is not a sufficient condition for P3.

The third one is a combination of both, the strategy can use the deductive reasoning point of view and the logical one. That can possibly call into question the formal logic point of view.

Besides, to refute the argument of Y, properties of sets might be used. Our hypothesis is that some easy counter-examples (like: in quadrilaterals, to have three 90 degrees angles is equivalent to have four 90 degrees angles) will convince more easily than theoretical speech which will thus be less likely to appear.

DV4: value of truth of the starting assertion

The assertion “A necessary and sufficient condition is that the diagonals cut in their middle” is false in the set of quadrilaterals but true in the subset of non crossed

quadrilaterals. Our hypothesis is that it compells students to take into account crossed quadrilaterals as soon as they leave the strict framework of formal logic. Crossed quadrilaterals are hardly ever taught in France, and we have shown in a previous experimentation that their presence enhance strategies based on sets point of view.

DV5: value of truth of the assumption of X

Our hypothesis is that, in order to confront the formal logic point of view and the deductive reasoning point of view, it is necessary that they give conflicting answers. Indeed, if both points of view say that X is right, then no discussion can take place. This is why we chose that the assumption of X is false from the deductive point of view whereas this argument could be valid from the formal logic point of view.

SOME RESULTS

Whereas the study of this experimentation is not finished, we can present right now some results and transcripts.

First of all, we can assert that a work on reasoning and implication was done. Indeed, no group found the problem obvious, they have all studied it during a long time. But, on the other hand, no group was stopped by difficulties of a mathematical nature. There were discussions, even though the mathematical objects were very well known.

> Robert: It is an exercise which, as a teacher, I would not give before university.

There was really a confrontation between the formal logic point of view and the deductive reasoning point of view in a lot of groups. In most of these groups, even when they agreed the logical answer, they used the mathematical properties to search whether X is right or not.

> Paul : Y says that if a condition is removed, it is not necessary any more, I do say it can remain necessary if it is already checked in the hypotheses.

But the group do not agree this ensemblist argument and keep searching implications between properties. Few minutes later, Paul convince them with a counter-example:

> Paul: In the parallelograms, the condition “four 90 degrees angles” is necessary to be a rectangle but the condition “one 90 degrees angle” is necessary too.

This argument accepted, the group wants to know if X is right. After a few counter examples, they conclude showing that they distinguished well the logical reflexion and the mathematical reflexion.

> Robert: X is wrong but the argument from Y is false.

Finally, we have seen marks of the three points of view. The sets point of view was also used in this task as shows Armelle’s argument.

To show to her group that Y is wrong, Armelle draws three “potatoes” (i.e. sets), *P*, *C1*, *C2* so that, in the set *P*, the sets *C1* and *C2* are equal, whereas outside *P* they are different. This is a theoric counter-example to the affirmation from Y, yet the group accepts it only at the last minute.

CONCLUSION

The analysis of the students' answers is still in progress. However, we can already say that the exercise fulfiled its role, as for the work on the implication since all groups have worked at least one hour on this problem. In addition, the three different points of view appear, implicitly or explicitly, in most groups. In particular, the sets point of view, which is not taught in france, appears many times. This is why we can do the hypothesis that the work, on this point of view, made at the time of the preceding meeting was used again.

These results are to be placed among others. Indeed, this problem forms part of a six hour experimentation on implication and reasoning. It includes other stages of work, in particular, other studies of written proofs, one problem of geometry and one problem in discrete mathematics. Moreover, this experimentation takes sense when one knows that it was preceded by two others, carried out in 1999 and 2000.This problem is, thus, to consider as part of a broader context.

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[1] Institut Universitaire de Formation des Maîtres (University Institute for the Formation of the Teachers)

[2] There were two other questions : (P1) two other parallel sides ? (P2) two 90 degrees angles ?

[3] not necessarily the same one as the two precedents, this was specified orally.

[4] a condition A is weaker than a condition B if the implication B $\Rightarrow$ A is true, i.e. if the set linked to the property B is included in the set linked to the property A. This expression is commonly used in french mathematics.

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FROM SINGLE BELIEFS TO BELIEF SYSTEMS: A NEW OBSERVATIONAL TOOL

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Two of the greatest problems of research on affective factors, and in particular, research on beliefs, is what *and* how *we observe. The first difficulty is due to the lack of a clear terminology; but even once it has been clearly decided what to observe, it is not easy to put this into practice. This report describes from a theoretical point of view the results obtained using a new questionnaire appositely designed to overcome some critical points of beliefs' observation.*

INTRODUCTION

The recent book on beliefs (Leder, Pehkonen & Toerner, 2002) shows a growing interest for this construct in mathematics education. The contributions underline the most important problems of research in this field: on the one hand the lack of an agreement on terminology (see Furinghetti & Pehkonen, 2002; Op't Eynde, de Corte & Verschaffel, 2002), on the other the difficulty in designing efficient observational tools (see Leder & Forgasz, 2002).

The problem of observing pupils' conceptions is common in mathematics education research; as Balacheff claims (1990, p.262):

> It is not possible to make a direct observation of pupils' conceptions related to a given mathematical concept; one can only infer them from the observation of pupils' behaviors in specific tasks, which is one of the more difficult methodological problems we have to face.

As concerns research about affective factors, and in particular the problems related to their observation, McLeod (1992) underlines the need of a multiple approach which alternates qualitative methods (such as interviews and direct observations) to quantitative ones (such as standard questionnaires like Likert scales) and Schoenfeld (1992, p.364) claims that:

> The older measurement tools and concepts found in the affective literature are simply inadequate.

The observational tools used in most research can substantially be grouped under five typologies (see McLeod D. & S., 2002): a) physiological measures, b) interviews, c) direct observation of subject, d) diaries, essays, etc. e) questionnaires.

Questionnaires have been proposed in different typologies (see Leder, 1985) and certainly are the most used instruments because they are easy to construct, administer and score. But, during the last years, the limits of questionnaires that ask students their agreement with certain opinions have been clearly highlighted.

First of all, with this method the beliefs that *the researcher* considers important (Munby, 1984; Eagly & Chaiken, 1998) are selected *a priori*: instead, *open* tests, such as essays or interviews, are much more effective in this respect. Moreover,

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respondents generally answer these questionnaires in a context and with goals that are different to those experienced when they deal with mathematics. In these cases, the well-known difference (Schoenfeld, 1989) between *beliefs expoused* and *beliefs in action* is highlighted: sometimes they are contradictory beliefs which respond to different goals and appear in different contexts (see also Cobb, 1986), i.e. sets of beliefs grouped in separate non-interacting clusters. Besides, it is often arbitrarily assumed that certain beliefs elicit in all individuals the same emotion (see Di Martino & Zan, 2003): respondents to questionnaires are requested to express their agreement with a certain statement and their emotional disposition (like or dislike) toward the statement is inferred from the grade of agreement. But as Green underlines (1971, p.42):

> Whenever a person holds a certain belief, he must also take some attitude[1] towards that belief; and that attitude is always itself capable of formulation as a belief. It is a belief about belief.

Last but not least, the most widely used questionnaires simply make a list of commonly held beliefs without considering the connection among the beliefs. On the contrary it is fundamental to consider the structure of belief systems[2] (i.e. not only the content of beliefs but also the way people held it) because taking into account the psychological strength of beliefs can help both overcome the mismatch between beliefs expoused and beliefs in action, and in the attempt to change beliefs (Cooney, 1993).

If the use of questionnaires is criticizable for the reasons discussed above, it is undeniable that the administration and the analysis of questionnaires require less time than those of interviews or direct observations and, at the same time, allow the collection of data having a higher statistical relevance.

In a three-years Italian Project (involving many researchers) about the evolution of attitude towards mathematics (*Negative attitude towards mathematics: analysis of an alarming phenomenon for the culture in the new millennium)*, in addition to essays, interviews, direct observations and Likert scales, a questionnaire called Integrated Questionnaire on Beliefs (IQB) has been specifically designed. IQB attempts to take into account some of the criticisms to questionnaires while maintaining some of their positive features. Although the strength of the project lies in the possibility of analyzing jointly the results obtained with different instruments, this report focuses on the discussion of the results of IQB from a theoretical point of view.

METHOD

The analysis of two questionnaires used in a previous study (Di Martino & Zan, 2002, 2003) suggested us the idea for a new questionnaire (IQB). In particular we wanted to take in account the complexity of the relationships among beliefs (belief

[1] He identifies attitude with emotional disposition.

[2] Green (1971) underlines three features of belief systems: quasi-logical structure (beliefs can be primary or derivative), psychologically centrality (some beliefs are more important to people than other), clusters'isolation (sets of beliefs can be protected from any relationship with other sets of beliefs).

systems) and between beliefs and emotions. The conclusions of that study were that a single belief can be linked to different beliefs in different individuals, i.e. can belong to different belief systems, and that the same belief can elicit in different individuals different emotions. We suggested the hypothesis that the emotion elicited by the given belief is not always *simply* linked to the belief itself, but to the interaction among beliefs in the cluster containing it.

We have chosen a belief (from now onwards it will be called statement A) from a list of 12 beliefs used in our previous studies:

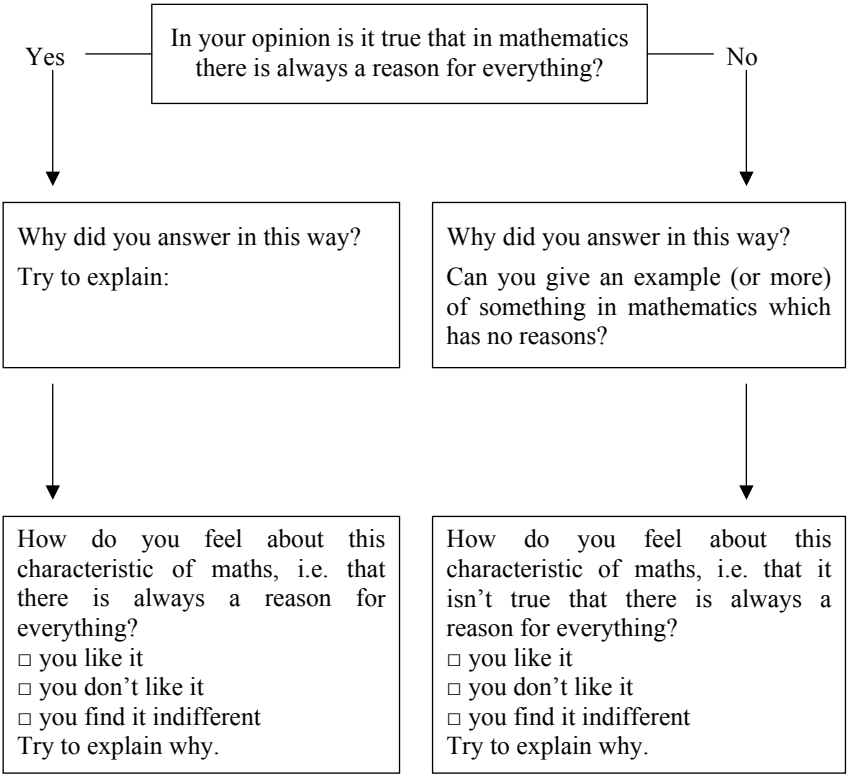
In mathematics there is always a reason for everything

The reason for the choice of statement A lies in the fact that statement A was recurrent in children's essays collected for previous research (obviously IQB's schema can be re-proposed with the choice of another belief).

Using statement A we have planned the following questionnaire:

Choose the answer (Y/N) that you most agree with.

Then follow ONLY the path signed by the arrow, and answer the subsequent questions.



IQB has been distributed in 13 classes of state middle schools (8 classes) and high schools (5 classes) and we have collected 282 questionnaires (178 from middle school and 104 from high school).

RESULTS AND DISCUSSION

The analysis of IQB has been very useful both to observe students' beliefs (together with the analysis of the results obtained with the other instruments used in the project) and for the ongoing construction of different profiles.
The numerical data are summarized in the following table:

	Yes	No
Middle school	170 (104 L-26 D-40 I)	8 (7 D-1 I)
High school	83 (46 L-12 D- 25 I)	21 (7 L-10 D-4 I)
Total	253 (150 L-38 D-65 I)	29 (7 L-17 D-5 I)

L = Like, D = Dislike, I = Indifferent

First of all we can observe that all the possible profiles (i.e. Yes-Like, Yes-Dislike, Yes-indifferent, No-Like, No-Dislike, No-Indifferent) are present, even if the percentages are considerably different. This is a further confirmation that it is arbitrary to link a fixed emotional disposition to a certain belief: for more than 40% of the students the agreement with statement A does not correspond to a positive emotional disposition and for approximately 24% of the students the disagreement with A is associated with a positive emotional disposition.
Another evidence is the high number of agreements to the statement A, but the difference in the percentages between high school and middle school students is significant: about 95.5% in middle school and about 80% in high school. Probably the greater *autonomy* of older students, who are less interested in the search for a *right* answer (i.e. an answer that they think the teacher would appreciate), can explain this difference in the percentages. A very important point for the interpretation of results is that in IQB respondents have to justify their agreement or disagreement. The analysis of the justifications shows that some answers of middle school students are influenced by commonplaces or by the belief that there is a *right* answer:[3]

6th Grade: I think so because mathematics is not an opinion[4]

7th Grade: Because mathematics is an exact science

[3] Some idiomatic expressions may be lost in the translation of the transcripts.
[4] It is an idiomatic expression in Italian meaning that mathematics is an exact science and therefore there is no room for autonomous ideas.

7th Grade: Since mathematics is thinking and logics there is a reason for everything

6th Grade: I think it is the most correct answer

8th Grade: I answered this way because I have heard that it is true

Other answers are based on personal school experiences and the focus is on mathematics as a subject matter instead of as a science:

6th Grade: I answered yes because I have always met problems with explained results. As in the case of the infinite number of numbers, this is possible because one can always add one to any number

11th Grade: I answered no because we have been taught many things without a clear explanation, they are that way and that is it

9th Grade: I answered no because sometimes we use methods which we have been taught since elementary school without an explanation of why things are so

6th Grade: I answered yes because the problems I have faced up until now have always had an answer to them

The justifications to disagreement are particularly interesting above all for the request to produce a mathematical example that supports their disagreement with statement A. We can note students' doubts and curiosities and personal epistemologies:

12th Grade: Why is $\infty \bullet 0$ not 0? Since elementary school we have been taught that any quantity multiplied 0 is 0. Now we discover that it is not true, why?

9th Grade: I do not understand why a number times zero is equal to zero and not to the number itself. As a matter of fact if I have a thing and I multiply it by nothing (zero) I still have that thing, it does not disappear!

6th Grade: For example: why is 2+2 equal to 4 and not 3 or something else

8th Grade: Why are numbers infinite?

11th Grade: An example is the association between numbers and names or figures. Why do we associate the word 'three' to the concept of three and why do we represent this concept with the symbol 3 and not 7?

Also in the case of agreement with statement A the analysis of the justification is meaningful because it can highlight belief systems linked to the agreement with statement A:

6th Grade: I answered yes because mathematics is difficult to understand so there must be an answer to all questions

11th Grade: Mathematics is the science of certainties: even situations which apparently are unreal have a proof

9th Grade: Because mathematics is based on explanations (although they are useless since I do not understand anything)

Moreover, the request to justify both the agreement with statement A and, using Green's terminology, the attitude about it can allow to single out some different interpretations of the statement: these *misunderstandings* are not rare and it is important to recognize them in order to interpret the results and to improve the instrument.

In the case of statement A it seems to us that some students have expressed their agreement/disagreement towards another statement: *In mathematics there are many*

open questions without an answer (obviously this misunderstanding depends on the language in which the questionnaires are written: in the case of IQB the language is Italian):

6th Grade: I answered yes because mathematics still has many secrets

Some students have expressed their disagreement with examples out of the mathematics context:

6th Grade: For example mathematics does not have answer to: 'does space have an ending?'

In the case of the emotional disposition associated with the agreement/disagreement on statement A some students have simply described their emotional disposition towards mathematics: for example *I like mathematics*. Probably this can be due to a careless reading of the request, nevertheless it is important, as before, to recognize the possibility that someone answers to a different question.

But the most important characteristic of IQB is the request to justify the emotional disposition associated with the agreement/disagreement to the statement A. As we discussed earlier, previous studies (see Zan & Di Martino 2003) suggested that the emotional disposition associated to a certain belief is not directly linked to that single belief but to a belief system containing it. In the design of IQB we hypothesized that asking respondents to justify the emotional disposition could give further information about their beliefs linked to the original belief (i.e. the belief that it is/is not true statement A) or about their personal epistemology:

8th Grade (YES-L): I like it because if we follow the rules we cannot fail

12th Grade (YES-I): It is indifferent to me because if my solutions to the exercises are correct I do not care if there is another explanation

8th Grade (YES-D): I do not like it because I like mysteries and unsolvable enigmas. In mathematics everything has an explanation, it is a boring world for boring people

9th Grade (NO-L): I like the fact that in mathematics not everything has an explanation because it makes it more intriguing. Besides this stimulates the desire to discover new things

12th Grade (NO-I): It is indifferent to me because I am not a mathematician

9th Grade (NO-D): I do not like it because it is difficult to learn rules by heart which we do not understand or which we cannot find with reasoning

The request to justify the attitude highlights that there are deeply different motivations for answering *indifferent,* these typologies are not so evident using other instruments as semantic differential scales or Likert scales.

In fact the answer *indifferent* can derive from:

a) A like or a dislike not too marked (this is the case typically considered with semantic differential scales):

9th Grade: It is indifferent to me, not really, I mean I like the fact of getting lots of answer but I am not so enthusiastic about it

b) An alternation between like and dislike depending on the contexts:

8th Grade: It is indifferent to me because I like finding answers, but not to everything (…) I do not like it when the teacher asks me to explain everything I have said

c) The belief that a personal opinion, whether right or wrong, cannot change anything:

11th Grade: It is indifferent to me because mathematics is that way, and even if statement A were false, it would be that way the same

d) The difficulty in recognizing and expressing the personal attitude towards a certain belief:

8th Grade: I have never thought about it enough and out of the blue I do not know if I like or not

CONCLUSIONS

Many researchers have underlined the importance of a discussion about instruments to observe beliefs. Obviously the development of beliefs'theory contributes to this debate: in particular the increasing relevance given to belief systems rather than to single belief *forces* researchers to construct adequate instruments.

IQB was created to meet this need and to preserve some of questionnaires'positive characteristics (like the easiness of administring and analyzing them).

The request of IQB is not only to express the agreement/disagreement with the statement A but also to motivate it.

But the greatest innovation of IQB is the request of expressing the reasons of one's emotional disposition toward the declared belief.

This allows the researcher to highlight other beliefs linked to the declared belief and the psychological centrality of the declared belief, thus giving information about the belief system containing it.

The results obtained when first using IQB are encouraging: analyzing the answers to IQB we found links between the agreement/disagreement with statement A and other beliefs.

The next step is to experiment IQB's schema with other statements: like those typically used in beliefs'research.

Obviously IQB can be improved and some changes have already been made. But above all, other instruments may be constructed. As a matter of fact we believe it is important to work in this direction both to improve the consistency of the instruments with the theory and, to interpret and compare the results obtained, in this field, up until now at best.

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ASSESSMENT AS A STRATEGIC TOOL FOR ENHANCING LEARNING IN TEACHER EDUCATION: A CASE STUDY

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This small exploratory case study describes an attempt to integrate the academic and practical aspects of a teacher education course in order to promote deep understanding of children's ways of understanding mathematics. The assessment regime of the course was used as a strategic tool for engaging students, and the assessment tasks themselves were used as the means of generating genuine integration, or case knowledge, of the content of the course. The results indicate that the approach was effective in achieving the aims of the course, and student reaction to the approach was extremely positive.

INTRODUCTION

It has been argued that integrating the academic and practical aspects of teacher-education courses can promote more effective learning by children (Even, 1999). That is, teachers who have had the opportunity to make meaningful connections between research and their classroom-based experiences, develop deeper understanding of children's ways of thinking about their mathematics. In Even's (1999) study, teachers conducted a mini research project as part of a professional development program, and their written reports of the research project helped them to reflect on their experiences, generating "genuine integration of knowledge learned in the academy and that learned in practice" (p. 250). These two forms of knowledge have been defined by Shulman (1986) as propositional or declarative knowledge that is hard to be applied and used, and case knowledge that makes propositions real, and embeds them in context:

> Case knowledge is knowledge of the specific, well documented and richly described events. Whereas cases themselves are reports of events, the knowledge they represent is what makes them cases. The cases may be examples of specific instances of practice — detailed descriptions of how an instructional event occurred — complete with particulars of contexts, thought and feelings. (p.11)

It is the integration of these two forms of knowledge that we want our teacher education students to achieve.

McInnis and Devlin (2002) state that good assessment at the tertiary level has three objectives:

1. It guides and encourages effective approaches to learning
2. It validly and reliably measures expected learning outcomes
3. It defines and protects academic standards.

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We believe that if the emphasis of the assessment regime is on the construction of case knowledge rather than on summative or grading practices only, then assessment can be used as a strategic tool for helping students engage with a course in a meaningful and integrated way, We argue that this focus constitutes good assessment practice at any level, including tertiary education.

In this paper we present a description of a small exploration of an approach that parallels Even's (1999) study, and in addition, uses assessment as the strategic tool for promoting more effective learning as suggested by McInnis and Devlin's (2002) first objective.

BACKGROUND

This paper describes an innovative approach assessment used in the first mathematics education unit undertaken by primary teacher education students. This one semester unit, *Children and Mathematics: Developing Numeracy Concepts*, has a focus on the early years of school, and aims to "promote students' understanding of how children's mathematical concepts develop … in number and measurement" (Deakin University, 2003, p. 351). The unit provides students with the opportunity to engage with young children, examine their mathematical developmental, and consider ways of providing effective learning experiences.

The unit content is presented in lectures that include relevant video excerpts and discussions led by mathematics education staff. The lectures are supplemented by tutorials in which students engage in practical tasks and discussion related to the content of the lectures. The assessment tasks for the unit are a team-based written report on the analysis of children's responses to a mathematics interview, an individual response to providing appropriate learning experiences for children, and a written examination on both the content and pedagogical knowledge presented during the unit.

An assessment task used for many years in a similar course was a student interview of two four- or five-year-old children about their number development using an interview that included a Piagetian number conservation task. A written report of the analysis of the children's responses to this interview formed part of the assessment requirements, while a verbal report on the interview tasks and findings was presented in tutorials, with discussion focused on interesting similarities and differences in the results.

The strength of this assessment task was that it demanded the integration of the academy (the lectures and tutorial content) and the practical (the interviews with children), although the extent of the integration was bounded by the students' engagement with the written report and their participation in follow-up class discussion. The weakness in this task lies not in the task itself, but in its relationship with the other academic and practical aspects of the unit content. For example, later content examined children's numerical development in more sophisticated aspects of mathematics such as operations with numbers and algorithms.

SHIFTING THE FOCUS

In 2003 for the first time this unit *Children and Mathematics: Developing Numeracy Concepts* was provided for 180 primary teacher education students on the Melbourne campus of Deakin University as well as to students who were attending other campuses of the university. This paper refers only to the implementation of this unit on the Melbourne campus.

As part of the development of this unit, the nature and role of assessment tasks were designed to act as a strategic tool for enhanced learning. An examination remained, but an interview and team-based written report were combined with the individual written description of appropriate learning experiences. The first task, the interview, was to take place while students were on a practicum placement in a primary school. As in previous years the interview had a focus on number. While we are aware of the difficulties for an untrained teacher when acting as a clinician, the highly structured interview protocol and its response format was considered robust enough to generate reliable data for the purposes of this assessment task (see, for example, Hunting & Doig, 1997; Haydar, 2003, on the value of training in clinical interviewing).

The first assessment task

The outline of the first assessment task to the students was similar to the description that follows:

> This is a team-based assignment with a focus on children's number development. You will form a Team that consists of four students, where one member of the Team will interview at each of the four year levels (Prep, 1, 2, or 3). Each member of the team will conduct an interview with two children from the same year level. The Team will thus have interview records from two children from four year-levels, a total of eight interview records. Teams will write a team report that includes an analysis and discussion of the development of children's number understandings as evidenced by the data that they have gathered across the year-levels.
>
> The Team's data and report will be entered on to a database *via* a web site. The contents of the database will be available to students for use in the second assessment.
>
> The Team report must indicate what the Team considers to be the main findings of their analysis of their combined data; and the implications of their findings. Team reports must be linked to the evidence gathered.

There are two points to note here: first, the focus is on the team and the analysis and discussion of the team's data, not the individual student's data. This places responsibility on students for conducting interviews and reporting accurately. Secondly, the use of the combined data allows the students to examine the development of children's mathematics over four years and not, as previously, within a single year level. This mirrors the academic content presented later in the unit that examines children's longer-term development. The requirement that the team's report

focus on the implications of the findings was a further step towards integration of the academic and the practical.

The format of the team reports was a poster that had a printed copy of the data attached, thus allowing a reader to see a reduced version of all the data, grouped into eight themes by content type (for example, numeral recognition, place value), together with written comments highlighting the implications of the findings. The posters were displayed for all students to read and follow-up discussions were held in tutorials, using the posters shown below in Figure 1 as aids.

Figure 1: Team posters reporting interview data and analysis

We agree with Crespo and Nicol (2003) that discussion springing from student-teacher interviewing provides insights into their underlying beliefs about teaching. For example, during follow-up discussions in tutorials some students were surprised to hear that there were children who were able to respond correctly to items beyond those expected by the curriculum. The response of some of these students was that the teachers, or parents, were ‘pushing the children’, the implication being that this was not a good practice. Other students responded quite differently, suggesting that these ‘advanced’ children must be attending private (non-government) schools, apparently implying that these schools were also ‘pushing’ the children. Other students suggested that as the children could respond correctly, then the mathematics was not beyond them at all. The discussion regarding this ‘pushing’ was lively, and students were able to draw on the data collected by themselves and their peers to support their points of view.

Tutorial discussion also raised issues that sparked interest. For example, a common finding in the data of many groups was that the Year 1 children frequently achieved

better than the Year 2 children. As this appears to be the case for children from different schools and areas of the city, this was seen as a real trend and generated many hypotheses as to its likely cause.

The tutorial discussions also revealed some of the students' own problems in mathematics. For example, discussion of the question "Which is larger, –7 or –4?" revealed that for some students the value of the digit identified the correct answer. During discussion it became apparent that students' analogies for working with negative numbers, as learnt in school, were sometimes misleading or erroneous.

A second point to note in the description of this assessment task is the requirement that children's responses to the interviews be entered onto a data-base for use in the second assessment task. This requirement was facilitated by two aspects of the poster format of the report. First, all interview items and (where applicable) correct responses were available to the students in electronic format for entering children's responses and printing a copy to form part of the poster.

Thus the entire data-set of responses from approximately 360 children, 90 at each of the four levels Prep, Years 1, 2, and 3, were available to one of the authors, who randomly selected sets of 30 children's responses to form four virtual classes (a Prep, a Year 1, a Year 2, and a Year 3). These four virtual classes provided the following details for each child in the virtual class: a pseudonym, sex, age, and all their interview responses. These virtual class details were provided on a web-site for student access. Down-loading their chosen class gave students access to the data on every child in the selected virtual class as collected by the students in their first assessment task.

The second assessment task

The second assessment task built upon the first in two distinct ways. The obvious way is the use of the virtual classes, based on the responses from the interviews of the first assessment task, as described. The second, less obvious, way, is that it required students to use their knowledge of children's mathematical understandings across year levels developed by the first assessment task, integrate it with their understanding of the content of the lectures and tutorials during the unit, and apply the resulting case knowledge. That is, the integration of the academic knowledge with the practical experience is embedded within the second assessment task by the requirement that students address the mathematical needs of children within a selected virtual class and adopt the role of a teacher of real children who have real mathematical needs.

The details of the second assessment task were

> This assignment is meant to give you a taste of creating focused, appropriate learning experiences for a whole class, or a small group within the class. There are several options for you to choose from, and these are set out below. First you must select the year level that you wish to have as your virtual class. Perhaps the level that you

worked in during your placement could make life easier, as the context may be more familiar to you.

Remember that the interview tasks run beyond what we would reasonably expect of Year 3s in order to 'capture' those children who are working beyond the usual. So, you

are not expected to plan experiences to cover all the interview tasks, simply those you consider to be most critical for your children.

Option 1

Look at your class. Think about whether there is an identifiable group of 4 or 5 children with a similar mathematical need; and if there is, you may want to focus on that need for these children. You must select and describe 3 to 4 tasks addressing the mathematical needs of this small group if you choose this option.

Option 2

Look at the range of abilities, described by the responses to the interview tasks, across the whole class. Look at the least capable and the most capable children. Can you plan a single experience for this 'spread' of capabilities? For this option, you will need to select and describe one task addressing the mathematical needs of all the children. The task should be one that the least able children can tackle successfully, but also one that is open to extension for the more capable in the class; this sort of task is often termed a 'ramped' task as it goes 'up' in difficulty and the children are able to continue exploring it as far as their ability will allow.

Specific questions

Once you have the topic and the children sorted out you should address the questions below to complete the main body of your assignment. The questions to answer are:

1. What are the mathematical learning needs of your selected children?
2. What tasks were surveyed, and from what sources?
3. What mathematics do the selected task(s) deliver?
4. How will you know if the tasks have achieved your aims?
5. What have you learned from this assignment?

Student responses to this second assessment task were, as expected, mainly positive with some themes evident across many of the responses to a particular question. These themes related particularly to Questions 2, 4, and 5.

The most common sources of information for responding to Question 2 were educational Internet sites. Many of these sites were North American in origin, and students modified and adapted the information to suit local curriculum and conventions. It appears that the Internet is replacing the photo-copiable worksheet book as a major resource in primary classrooms. Another source, that was cited frequently, was the teacher in the class where the first assessment task (the numeracy interview) had taken place.

Surprisingly, responses to Question 4, that focused on assessing the effectiveness of one's teaching, seldom involved re-using the interview items that had revealed the original strength or weakness. Most responses considered that success on the selected 'new' task was sufficient to establish effectiveness. While this is in part true, the re-use of the interview items would seem a more reliable and valid approach to this question.

The final question of this assessment task was designed to provide students with a space in which to reflect on their experience and there were strong themes in the responses. One of these themes centred around the reality of the task, with this task compared favourably to other assessment tasks in the students' course that were considered not as relevant to their futures as teachers. Another common theme was the students' realization of the difficulty of finding and selecting tasks suitable to address particular needs. Comments focused on the length of time needed to find and select such tasks, as well as the time needed to unpack the mathematical content of many tasks. Comments on the low quality of many Internet sites were also common.

Student reactions to the two assessment tasks overall were very positive. Comments from students revealed that the workload was reasonable, the use of groups and posters was an engaging way to respond to tasks, and that re-using the interview data was a sensible and useful exercise. In particular, students commented on the re-use of the data as providing a familiarity that made the second assessment task less daunting, although many students wondered whether they would have enough time to do this type of task properly in a real classroom.

DISCUSSION

There are many facets to this exploratory case study but two aspects are of most interest here. The first is the effects of integrating the academic and practical aspects of a teacher education course in order to promote deep understanding of children's ways of understanding mathematics. The modifications to the assessment regime of the course were made in order to ensure that assessment would act as a strategic tool for engaging students, and promote such integration as an integral part of the course. This was accomplished by the two-fold use of the same data: in the first instance for examining children's development across the years, and in the second, to conduct a more detailed examination of the variation in children's needs at one year level.

The success of this strategy can be established by reference to student responses to the assessment tasks themselves, and their comments in their unit evaluation surveys (an obligatory part of the teaching process). While many of the comments were typical of student comments everywhere, with complaints about early morning lectures, too much content to be learned, and praise for particular aspects of the unit, a large number of comments were related to the professional aspects of their experience in this unit. These comments focused on those aspects of the unit that we believe were critical to achieving our aim of integrating the academic

and practical, and that built a strong relationship, between the unit content and assessment, and the students' own case knowledge and professional preparation. We believe that such integration of the different knowledge forms is a basis for a teacher's professional practice and should be the aim of all teacher education courses

Students' reflective comments indicated that there was sufficient evidence for continuing to use assessment as a strategic tool for integrating the academic and the practical, and that these students were building a reflective approach to teaching mathematics likely to promote effective mathematical learning experiences for children..

A further outcome of this study is that it raises the issue of Shulman's case knowledge and its place in teacher-education. Clearly case knowledge represents an ideal for student outcomes in this context, and the attempt described here shows a possible way forward in achieving this goal. It is hoped that the more detailed study underway at present will reveal more clearly those features of assessment as a strategic tool that are critical to achieving our goals.

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PATTERNS OF REASONING IN CLASSROOM

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In this paper we report on some patterns of reasoning, which emerged during an activity of proving a mathematical statement performed by nine grade and university mathematics students. The statement in question involves drawing figures, working in arithmetic and in algebra. As for secondary students we detected fluency, flexibility and ability of verbalizing their reasoning. In particular, we will focus on the behavior of a student who through drawings succeeded in giving meaning to algebraic manipulation. The solutions of the university students were conditioned by the burden of the formal style used in university course of mathematics.

INTRODUCTION AND THEORETICAL FRAME

In works presented in PME meetings, see (Furinghetti & Paola, 2003), and in journals, see (Furinghetti, Olivero & Paola, 2001) we have focused on patterns of reasoning emerging when students are involved in activities of exploring, producing and validating conjectures. To rouse motivation and non-routine behaviors we set these activities in stimulating contexts such as group working, classroom discussion, and use of technology. In the present paper we go on in our investigation on patterns of reasoning by studying the results of an experiment in which proof was proposed as an intellectual challenge. In history this challenge has been fundamental for producing of new mathematical ideas and some authors, (De Villiers, 1996) for one, think that even nowadays it may be a motivation in classroom. The aim of the study is twofold: • as researchers in math education to collect information about students' reasoning, • as teachers or teachers educators to outline some didactical implications.

In our investigation we had in mind certain aspects of students' reasoning to be analyzed, which guided the choice of the statements to be proved. In the following we briefly discuss these aspects. One of the driving forces in performing mathematical tasks is *transformational reasoning*. According to (Simon, 1996) transformational reasoning is a third type of reasoning (beyond deduction and induction), which is not a mere gathering of information, but rather the development of a feeling for the mathematical situation a person is facing. It is the realization (physical or mental) of an operation or a set of operations on objects that brings to reconsider the transformations which the objects undergone to and the results of that operations. In transformational reasoning it is central the ability to consider not a static state, but rather a dynamic process in which a new state or a continuity of states are generated. Transformational reasoning is reasoning by analogy and anticipation. It may produce a different way of thinking to mathematical objects, as well as a different set of questions and problems. Transformational reasoning is enhanced by fluency and flexibility, that is to say the abilities to overcome fixations in mathematical situations and to produce creative thinking within mathematical

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situations, see (Haylock, 1987). Gray & Tall (1994) focus on the flexibility as a mean for linking processes and concepts.

Among the abilities necessary to mathematical activities Selden and Selden (1995) take into consideration concepts reformulation. When a statement to be proved is given, the solver firstly needs to understand it. This may happen through reformulating the statement by paraphrase with words, by gestures, by figures, by symbols, by the production of examples. Among proving difficulties Moore (1994) considers the generation and the use of examples. Examples may work as prototypes, but have not to become stereotypes, see (Presmeg, 1992). In the examples the solver has to develop a process of abstraction and generalization, that is, borrowing the expression from (Mason & Pimm, 1984) "seeing the general in the particular".

A student's behavior in proving may be analyzed according to the framework of proof schemes defined by Harel and Sowder (1998) as what constitutes ascertaining and persuading for a student. The proof schemes are grouped in three main classes:

• external conviction (three types: ritual, authoritarian, symbolic)
• empirical (two types: inductive, perceptual)
• analytical (two types of proof schemes: transformational, axiomatic).

A ritual proof scheme manifests itself in the behavior of judging mathematical arguments only on the basis of their surface appearance: false arguments are accepted because they look like usual proofs, and on the contrary justifications that are even convincing are rejected because "they don't look as mathematical proofs". In a symbolic scheme mathematical facts are proved using only symbolic reasoning, i.e. using symbols without reference to their meaning. An authoritarian proof scheme relies on the authority of someone (book, teacher). An inductive proof scheme relies on few examples without generalization. A perceptual proof scheme is based on rudimentary mental images without resorting to deduction. A transformational proof scheme encompasses a deductive process including generality, goal-oriented and anticipatory mental operations, and transformational images. In addition to that an axiomatic proof scheme contemplates the presence of an axiomatic system.

According to Rodd (2000, p.231) the following questions are crucial: "(b) What is the personal nature of proof? […] Or why are students' personal justifications different from the paradigmatic mathematical proof? […], And (c) What might warranting mean in classroom practice?" These issues are discussed in (Hanna, 1990; Hersh, 1993) as for students and mathematicians, Barbin (1988) as for the history.

METHOD

Our experiment took place in two different settings (setting **A**: secondary school, setting **B**: university). A set of problems centered on proof was given to students. The students knew that their performance would not be assessed with a mark. They were only asked to engage as much as possible in the solution of the problems and to write all their thoughts during the solution. We also asked them to write the difficulties

encountered and if they enjoyed the problems. All students signed with a pseudonym their protocols. In the present paper we will focus on the following problem:

> Given a cube made of little cubes all equal, take away a full column of little cubes. The number of the remaining little cubes is divisible by six. Try to explain why this happens.

It has had been chosen according to the following requirements:

- it is expressed by words
- it involves concepts that are at the grasp of the students
- it does not involves only rote manipulation, but rather requires to look at algebraic formulas with meaning and awareness
- even if the property to be proved is given, the form of the statement (which includes the invitation "Try to explain…") fosters exploratory activity and the devolution of the teacher's authority to the students, as it happens in the case of open problems
- the statement requires to consider different aspects: geometrical and visual (the cube formed by little cubes), the numerical aspects (divisibility by six), symbolic (the formulas which express the number of remaining cubes and the algebraic manipulation on them). Thus the students have to use different frames and to pass from a frame to another
- it is not similar to statements proved by the students in other circumstances, thus students are stimulated to find their own way.

The setting A (secondary students grade nine)

In secondary school 18 students of grade nine (aged about 14) faced the problem in question. They had worked before in collaborative groups and thus we allowed them to work in group. The students were used to be involved in activities of exploration, production and validation of conjectures. In particular, they were able to perform these activities with the symbolic pocket calculator. The classroom was really a *community of practice*, as advocated by (Schoenfeld, 1992). The allowed time was 90 minutes. In the first 15 minutes the students were asked to work individually on the problem before starting the work in group. This splitting in two phases was decided because it happens that without an initial phase of personal reflection the interaction in the group may be only apparent and some members of the group follow passively the solving strategies proposed by their mates. During the work the teacher and the observer were at disposal of students for giving explanations and to foster the exploration. At the moment of the experiment the students did not know the algebraic manipulation of formulas, but they had used regularly the symbolic calculator; they mastered the commands *Factor* and *Expand*. The command *Factor*, indeed, has been used before only for decomposing numbers, but it was easy to extend this use to algebraic formulas. To have at disposal the calculator allowed keeping the focus on the problem and not on 'side issues' such as algebraic manipulation. In our intentions the resulting atmosphere in the classroom should have been rather relaxed so that the moments of strong emotions for stops or failures should have been avoided or, at least, overcome through collaboration and communication. In this situation all students, even the weak ones, had the possibility of producing some materials.

The setting B (university students)

The five university students participating to the experiment were attending the third year of the mathematics course. They had already passed examinations such as analysis, algebra, geometry, and topology. They worked alone and did not interact with the university lecturer and the observer. The allowed time was 60 minutes for the problem in question and another problem that we do not consider here.

FINDINGS

The teacher or the lecturer (the authors D. P. in school and F. F. in university) together with an external observer (the author M. C.) assisted to the experiment. The data were collected through the students' protocols and the observer's field notes.

Findings in the setting A (secondary students grade nine)

We have at disposal 18 protocols coming from six groups. All groups, but one, reached the solution. In Fig.1 we report the cognitive pathway towards successful proof, which emerges from the protocols.

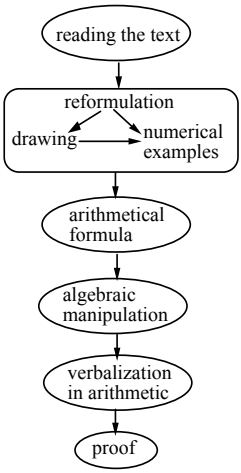

Fig.1

Each step of the pathway requires a shift from one frame to another. The word 'cube' in the statement pushed naturally towards the representation of a cube in the flat sheet according to empirical rules of perspective (graphical frame). Through the drawing the statement was reformulated in a more telling way. After the exploration of the drawing the students used it as a starting point for producing a few numerical examples (arithmetic frame). The drawing worked as a generic example that allowed to generalize and to produce the solving formula n^3-n. Borrowing the metaphor from (Tall & Gray, 1994) we may say that the drawing plays the role of the pivot between the particular (numerical examples) and the general (formulas). At this point the shift into the algebraic context allowed obtaining the decomposition $n(n-1)(n+1)$. The conclusion was reached by verbalizing the property that the product of three consecutive numbers is divisible by six.

To stress the importance and the peculiarity of the role plaid by the students' drawing we consider the work of a group of three boys (Andrea, Luca, Simone) in which an interesting process was produced. They started by drawing a cube. Firstly they explored a cube formed by 3^3 little cubes and went on by alternating exploration of inductive type (the cases of cubes formed by 4^3, 5^3,… little cubes) with reflections on the particular case of the cube they had drawn. The drawing acted as a generic example. The exploration of particular cases went on also after the determination of the formula n^3-n. The solving strategies were a continuous 'come and go' from

consideration of concrete situations (particular cubes and calculation on them) to reasoning on formulas and attempts to write them in different ways. In this phase the teacher acted in the proximal development zone of Vygotskij (1978). He asked to students which ideas they were relating to divisibility. Simone mentioned multiples, Luca and Andrea decomposition. The new idea of decomposing n^3-n came through a process by abduction, see (Otte, 1997). At this point the teacher suggested using the symbolic calculator to decompose the formula. Immediately after having obtained the decomposition $x(x-1)(x+1)$ the students verbalized the solution: "Given three consecutive numbers at least one is even and one is divisible by three". We note that the decomposition was written exactly as we reported (the name of the variable n was changed into x.) This was a spontaneous sign given by the students of their shift from the arithmetic to the algebraic frame.

Andrea, however, was not satisfied with this solution and looked for a different process. One of the reasons of this dissatisfaction could have been the fact that the solution was found through the teacher's intervention and thus Andrea felt that he was not controlling the situation and needed to 'take possession' of the solution. He reflected on his drawing and we saw him to make gestures by hands, to think intensely until he found a new solution, based on the decomposition and composition of the original cube until a parallelepiped was obtained, see Fig.2. The teacher asked Andrea to write how he reached the new solution and why he looked for it. He wrote (for the reader convenience we have translated):

> I was not satisfied at all with the decomposition made with the symbolic calculator (I was thinking: Why I have not suddenly thought to the factorization?) [He is referring to the fact that before decomposing n^3-n he worked a lot around the figure] and I was 'looking at' [The inverted commas are in Andrea's text] the figure, partly to see that 'monster' and partly because I wished to find a geometrical proof [Andrea tries to give meaning to what is doing. He seems disturbed by the contamination between the geometric context of the problem and algebra]. Rather unconsciously - may be by vent - I started to strike off the column in question. When I saw the column struck off I realized that the two remaining columns should have been moved so that a rectangle [he means, indeed, a parallelepiped] is formed, which is high a column less (x-1), deep equally (x), and large one column more (x+1). Since the formula which gives the volume of the rectangle [parallelepiped] is b•h•p, I wrote x(x-1)(x+1), which was the same to the factorization of the calculator. To better understand my idea see the sheet [Fig.2] with the steps of the operation.

The expression 'to look at' suggests that the student's behavior is guided by ways of thinking oriented to the production of a proof. The process carried out by Andrea is mainly based on transformational reasoning. This reasoning was enhanced by three different kinds of signs used in an integrated way. We know that Peirce distinguishes among three kinds of signs: - icon, i.e. something which designates an object on the ground of its similarity to it; - index, i.e. something which designates an object pointing to it in some way; - symbol, which designates an object on the ground of some convention. Andrea uses all these kinds of signs in an integrated way. Initially the icon (drawing) is the way of paraphrasing the problem. The gestures by hands are

a means to enhance transformational reasoning. In the very words written by the student ("I would have wished to find a solution only with numbers") we see that for him symbols hide meaning, while the drawing is a carrier of meaning. We note that the student operates on his drawing in a symbolic mode. He, indeed, manipulates the pieces of the cube as representatives of the algebraic symbols x, x-1, x+1.

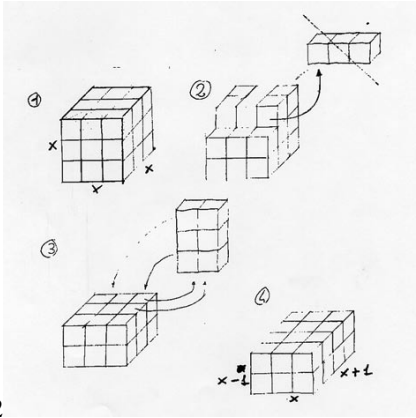

Fig.2

The first mode of solution produced by the group of Andrea may be ascribed to an axiomatic-like proof scheme (they 'derives' that the number of the remaining cubes is multiple of six), while the second mode enacted by Andrea alone belongs to the transformational proof scheme (he 'sees' that that number is multiple of six). The discrepancy of schemes shown by this student is an evidence of a discrepancy between proofs which prove and proofs which explain, see (Hanna, 1990). We found interesting that in the group the two mates of Andrea acted in a different way. They both worked only inside the algebraic frame asking for formal aspects and avoided reference to concrete situations.

x

10/4

Fig.3 x 10/4

The process conceived by Andrea has resemblance with the 'cut and paste' process realized by Al-Khwarizmi (1838) for solving second degree equations. In the case of the equation $x^2+10x=39$ Al-Khwarizmi starts from a square of side x, sticks on the four sides four rectangles of sides 10/4 and x. He obtains a cross (see Fig.3) whose area is x^2+10x (which is equal to 39). Four squares of side 10/4 are added to the cross to obtain the final square whose area $(x+10/2)^2$ is equal to $39+4(10/4)^2$. By equalizing

these quantities the usual solving formula for second degree equations follows. Al-Khwarizmi was interested only on positive solutions.

Findings in the setting B (university students)

We have at disposal five protocols. One student produced the solution in 10 minutes writing only seven lines. He did not draw any figure: he only reformulated the statement by words introducing the variable n for the number of the little cubes in the edge and then immediately generalizing the problem. Afterwards he wrote the solving formula n^3-n, decomposed it and through verbalization proved the divisibility by six. The rapidity of the succession of steps shows a strong anticipatory thinking.

The other four students followed a different pattern (more or less the same for all). They drew a cube with three little cubes in the edge and used it very easily as a generic example to produce the solving formula without the need of exploring other cubes or numerical examples. The divisibility by six was expressed by writing n^3-n=6q, q being a natural number. This formula is an example of formula which has not future, that is it is not "formally operable" in the sense of (Bills & Tall, 1998, p.105) since it is not easy to use it "in creating or (meaningfully) reproducing a formal argument". One student was suddenly discouraged and stopped after just one attempt. Other students went into the tunnel of the ritual proof scheme. They acknowledged the status of real proof only to proofs that appear as the usual proofs they have seen in university courses. For this reason they did not consider verbalization as a means for proving. This pattern of reasoning is clearly evidenced in the protocol of the student G. After having wrote the formula n^3-n=6q she decided to prove the statement by induction. We guess that this choice was inspired by the presence of the generic number n. She used properly the technique of induction and arrived at a statement requiring mere arithmetical considerations. At this point, since she had 'paid her debt' to the ritual aspect of formalism, she dared use verbalization (that she refused at the beginning) to conclude the proof.

DIDACTICAL IMPLICATIONS

The university students offer materials to answer the question "Why in education more does not always mean better?" The great amount of formal mathematical knowledge and the habit to use it as the only resource for doing mathematics has inhibited the ability to look for meaning in algebraic formulas. Our analysis of the secondary students' behavior has evidenced many aspects. Here we stress the fact that the message of the teacher had different outputs even when the conditions were the same. We owe the opportunity to grasp this fact to the style of teaching in the classroom where the experiment took place. As told before, only one in a group of three students adopted the 'cut and paste' method, his two mates preferred to look for a formal approach inside the algebraic frame. The filter of the individual's personality changes the way in which students perceive proof. The ascertainment of this fact brings to the fore the importance of studying the forms of classroom communication in relation with the different students' needs.

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GENERALIZED DIAGRAMS AS A TOOL FOR YOUNG CHILDREN'S PROBLEM SOLVING

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Measure Up is a research and development project that uses findings from Davydov (1975) and others to introduce mathematics through measurement and algebra in grades 1–3. This paper illustrates the use of generalized diagrams and symbols in solving word problems for a group of 10 children selected from a grade 3 Measure Up classroom. Students use the diagrams to help solve word problems by focusing on the broader structure rather than seeing each problem as an entity in and of itself. The type and sophistication of the diagram can be linked to Sfard's theory (1991 1995) on mathematical development. The consistent use of the diagrams is related to students' experience with simultaneous presentations of physical, diagrammatic, and symbolic representations used in Measure Up.

INTRODUCTION AND THEORETICAL FRAMEWORK

Solving number sentences (equations) and word problems that involve number sentences is an area in early grades that often creates much difficulty for children. This may be related to children's misconceptions about and misapplications of equations, including the use of the equals sign (Kieran, 1981, 1985, Vergnaud, 1985) or an inability to think beyond the literal word translations of the problem to see the more general structure. This phenomenon is likened to an intraoperational period (Garcia & Piaget, 1989) in that initially, students are solely concerned with finding a solution to a specific equation or problem and the equation is treated as an object that exists only to solve a particular problem.

If children, however, are to make sense of generalized statements and classify problems into larger groups to create more efficient and robust methods of solving them, then a different approach to introduce problem solving might be considered. One such approach stems from the work of V. V. Davydov and B. Elkonin (Davydov, 1975) and is embodied in Measure Up (MU), a project of the Curriculum Research & Development Group (CRDG) of the University of Hawai'i.

Davydov (1975a) believed that very young children should begin their mathematics learning with abstractions so that they could use formal abstractions in later school years and their thinking would develop in a way that could support and tolerate the capacity to deal with more complex mathematics. He (1975b) and others (Minskaya, 1975) felt that beginning with specific numbers (natural and counting) led to misconceptions and difficulties later on when students worked with rational and real numbers or algebra. He combined this idea with Vygotsky's distinction between spontaneous and scientific concepts (1978). Spontaneous or empirical concepts are developed when children can abstract properties from concrete experiences or instances. Scientific concepts, on the other hand, develop from formal experiences

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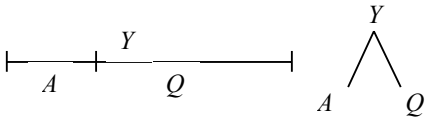
with properties themselves, progressing to identifying those properties in concrete instances. Spontaneous concepts progress from natural numbers to whole, rational, irrational, and finally real numbers, in a very specific sequence. Scientific concepts reverse this idea and focus on real numbers in the larger sense first, with specific cases found in natural, whole, rational, and irrational numbers at the same time. Davydov (1966) conjectured that a general-to-specific approach in the case of the scientific concept was much more conducive to student understanding than using the spontaneous concept approach.

Davydov (1966) wrote, "there is nothing about the intellectual capabilities of primary schoolchildren to hinder the algebraization of elementary mathematics. In fact, such an approach helps to bring and to increase these very capabilities children have for learning mathematics" (p. 202). Beginning with general quantities in an algebra context enhances children's abilities to apply those concepts to specific examples that use numbers.

Davydov (1975a) proposed starting children's mathematical experiences with basic conceptual ideas about mathematics and its structure, and then build number from there. Thus young children begin their mathematics program in grade 1 by describing and defining physical attributes of objects that can be compared. Davydov (1975a) advocated children begin in this way as a means of providing a context to explore relationships, both equal and unequal. Six-year-olds physically compare objects' attributes (length, area, volume, and mass), and describe those comparisons with relational statements like $H < B$, where H and B represent unspecified quantities being compared, not objects. The physical context of these explorations and means by which they are recorded, link measurement and algebra so that children develop meaning for statements they write and do not see them as abstract.

In the prenumeric phase, children grapple with how to make 1) unequal quantities equal or 2) equal quantities unequal by adding or subtracting an amount. A statement representing the action is written: if $H < B$, students could add to volume H or subtract from volume B. First graders observe that regardless of the action they choose, the amount added or subtracted is the same and is called the difference.

Number is introduced when students are presented with problematic situations that require quantification. It is then that a unit is presented within a measurement context. First graders re-examine some of the situations where they transformed two unequal quantities into equal amounts. For example, when working with mass, they may have written $Y = A + Q$. They notice now that mass Y is the whole and masses A and Q are the parts that make up the whole.



Both diagrams represent the relationships among the parts and whole of any quantity. From these diagrams children can write equations in a more formal way.

$$Y = A + Q \quad Q + A = Y \quad Y - Q = A \quad Y - A = Q$$

The part-whole concept and related diagrams help young children organize and structure their thinking when they are working with word or contextual problems. The scheme supports children writing equations and identifying which ones are helpful in solving for an unknown amount, without forcing a particular solution method. For example, students are given the following problem:

> Jarod's father gave him 14 pencils. Jarod lost some of those pencils, but still has 9 left. How many pencils did Jarod lose? (Dougherty et al., 2003)

In this case 14 is the whole, 9 is a part and the lost pencils (x) are a part. There are at least four equations (a fact team) that can be written to describe this relationship.

(1) $14 = 9 + x$ (2) $14 = x + 9$ (3) $14 - 9 = x$ (4) $14 - x = 9$

The third equation, $14 - 9 = x$, could be an appropriate choice to solve for the unknown. Some of the students in the MU research study use that method. However, other students choose to use the first or second equation to solve for the unknown amount. Their reasoning follows the compensation method for solving an equation by asking the question, "What do I add to 9 that gives 14?"

DESCRIPTION OF STUDY AND DATA COLLECTION

Measure Up uses design research (Shavelson, Phillips, Towne, & Feuer, 2003) as a means of linking research with the intricacies found in classrooms. The design research includes two sites 1) Education Laboratory School (ELS) Honolulu, HI and 2) Connections Public Charter School, (CPCS), Hilo, HI. These sites were carefully selected to provide a diverse student group representative of larger student populations in regard to 1) student performance levels, 2) socio-economic status, and 3) ethnicity. Student achievement levels range from the 5th to 99th percentile, with students from low to high socio-economic status and ethnicities including, but not limited to, Native Hawaiian, Pacific Islanders, African-American, Asian, Hispanic, and Caucasian. Students at ELS are chosen through a stratified random sampling approach based on achievement, ethnicity, and SES. No segregation or tracking of students is done at either site; all special education students are part of an inclusion program. Both sites have a stable student population for longitudinal study.

The MU project team is in the classroom observing and/or co-teaching daily. Three project staff members record observation notes in three separate formats. One person is responsible for documenting the mathematical development, one scripts the lesson, and one records instructional strategies used. Observation notes form a microgenetic study of student learning related to the materials under development. Videotapes are made of critical point lessons where the complexity of the mathematics shifts. Indicative of the design research approach, the MU project team discusses the types of problems or tasks that were used, the discourse that evolved from them, the

expectations about participation across the broad range of students, the role(s) played by representations and tools in the learning process, and the mathematics itself in debriefing sessions where analyses of data and interpretations of such are done.

Two types of individual student interviews are also conducted. One type, "teaching experiments," is adapted from a similar technique used in developing an algebra program at CRDG (Rachlin, Matsumoto, & Wada, 1987). As tasks are presented, students are asked to "think aloud" as they attempt the them. If students stop explaining, the interviewer prompts them by asking 'What are you trying to find (do)?' or 'What's giving you a problem?' If difficulty persists, the interviewer gives more specific clues, increasing the specificity until the child can complete it. The clues range from noting a particular error to more direct instruction.

In the second type of student interviews, project staff gives students tasks that would represent the level of mathematics that students would have experienced in a more conventional program or that focus on bigger ideas. Responses from students are compared to students from other grade levels that have not been part of MU. The purpose of these interviews is to determine how the mathematical understandings of students in the MU project compare to those students in other programs.

During student interviews, a group of mathematics educators, mathematicians, and psychologists has the opportunity to watch and participate in the interview live. The interviewer wears an earphone, and a video camera projects the interview into another room where the group is seated. The group can suggest additional questions through a microphone for the interviewer to pursue, enhancing the information gathered from the interview.

Types of problems presented and solutions used

As students move through their mathematical experiences, they use a variety of means of representing problem situations. They follow a progression similar to what Sfard (1991, 1995) described as a historical perspective (interiorization, condensation, and reification). As students work in each mathematical concept, *every* new idea is introduced with a physical model that is simultaneously represented with an intermediate model (like a diagram) and a symbolization (equation or inequality). This does not follow the typical approach where students work with physical models or manipulatives, *then* move to iconic or pictorial representations, and lastly, work only with symbols. The advantage of simultaneous representations is that students form a cohesive mapping of what the symbols represent and can see patterns and generalizations beyond the specific problem.

What follows are three snapshots of ten grade 3 students, taken from the 2003–04 school year. These students have been part of the Measure Up project since grade 1 with the exception of one student (Student S) who joined the class this year. In the first and last snapshots presented the data were taken from the classroom. The second example is from individual student interviews of these children. The tasks show the bridging from grade 1 and 2 mathematics to a more sophisticated level. Their

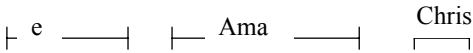
responses are linked to Sfard's progression (1991, 1995) such that an interiorization response is indicative of problem-specific characteristics; a condesation response indicates a transition from a specific to general solution approach, and a reification response embodies multiples forms of representation as a means of generalizing the structure of the problem.

Sample 1

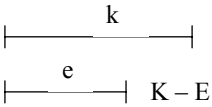
> Ama caught *k* fishes and Chris caught *e* fishes less. How many fishes did Chris catch?
>
> Dusty had *b* fishes in a bucket. When Anthony added his fish, there were *g* fishes in the bucket. How many fishes did Anthony catch?
>
> Sarah caught *p* fishes, and Tara caught *c* fishes. How many more fish did Sarah catch?

Students were asked to 'show the parts and whole in a diagram or on line segments' (Dougherty et al., 2003).

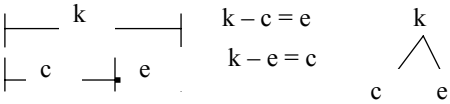
Students' representations varied with their ability to handle complex cognitive tasks. A sample of each type in response to the first problem follows.

Interiorization 

(Student M, 2003)

Condensation 

(Student S, 2003)

Reification 

(Student RI, 2003)

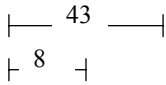
Sample 2

> Reed gave Jackie a strip of paper *w* length-units long. He gave Macy another strip of paper 9 length-units shorter than Jackie's. How long are Macy's and Jackie's lengths altogether?
>
> Jason has *v* mass-units of rice. Jon has *k* mass-units less than Jason. How many mass units of rice do Jason and Jon have together?
>
> Karyn had 43 volume-units of water in one container. In another container, she had 8 volume-units less than in the first container. If she pours all the water into one large container, how many volume-units of water will she have?

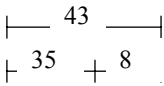
Students were asked to decide in what order they would like to solve the problems. Then, solve the problems in any way they wanted.

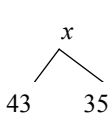
Consistently, regardless of cognitive level, students opted to do the numerical problem first. The mixed non-specified/numerical problem was chosen second, with

the non-specified problem chosen last. Four students indicated that it did not matter what order was chosen, they were all solved the same way. They did, however, proceed to solve the numerical one first. In all cases, students, including those who had used the interiorization method at the onset of the school year, used either a condensation or reification approach to solve the problems.

Condensation 

43 – 8 = 35 35 + 43 = 78 (Student C, 2003)

Reification 



$x = 43 + 35$ (Student RE, 2003)

$x = 78$

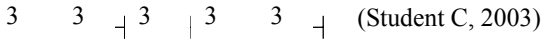
Sample 3

When his father takes 1 step, it takes Michael 3 steps to travel the same distance.

a. Michael's father walks 5 steps to get from the front door to the sidewalk. How many steps would Michael take to get from the front door to the sidewalk?

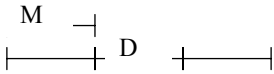
b. Michael took 27 steps to get from the front door to his neighbor's front door. How many steps would Michael's father take to walk the same distance?

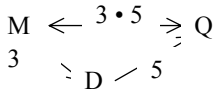
These problems were used in the introduction of multiplication. Without direct instruction, students used similar diagrams to represent, and assist in solving, the problems. Samples from each type are as follows.

Interiorization 

(Student C, 2003)

Condensation 
$3 + 3 + 3 + 3 + 3 = 15$ (Student J, 2003)

Reification 



$3 \bullet 5 = x$ $x = 15$ (Student M, 2003)

The generalization of the structure increases with each successive level. The diagram used in the reification approach is indicative of the general model of multiplication used in Measure Up. Unit *M* (in this case, Michael's step) is used three times to make intermediate unit *D* (the dad's step). Intermediate unit *D* is used five times to make quantity *Q*. If Unit *M* had been used by itself to create *Q*, *Q* is then represented as the product of 3 and 5. This last diagram embodies the use of a unit used to create a larger, intermediate unit as a means of creating a quantity. As MU students explore

multiplication and division, this model helps them define the quantities they are working with and their relationships.

IMPLICATIONS

Student solution methods strongly suggest that young children are capable of using algebraic symbols and generalized diagrams to solve problems. The diagrams and associated symbols can represent the structure of a mathematical situation and may be applied across a variety of settings. Students appear to utilize some form of the diagram and regardless of the sophistication of that model, students are developing a fluidity that allows them to attempt, and solve, word problems. The use of algebraic symbols and diagrams appears, at this stage of the research, to positively impact on students' mathematical development, especially when children develop their understanding of, and applications for, such diagrams through an approach that consistently and simultaneously links the physical model, intermediate representations, and symbolizations within each lesson, and not in a sequential manner.

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CORRESPONDENCES, FUNCTIONS AND ASSIGNATION RULES

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In this paper we put forward a theoretical position that, in cognitive terms, a differentiation should be made between a correspondence and a function. Important in understanding this difference is the role of an assignation rule; the correspondence acts as a way to identify a rule in context, whilst the function accommodates the rule in a more formal framework providing a secure base for argumentation. This perspective is used to interpret some students' behavior in a task where the identification of a particular relationship is crucial for its solution.

Introduction

When the word 'correspondence' is invoked in mathematics education literature, it is usually done in contrast with what is often termed 'covariation'. This 'duality' in the notion of function can be said to have its roots historically; mathematicians up to the 19th century usually handled perceived relationships as the co-ordination of variables, whereas latter mathematicians tended to insist on ordered pairs. Papers dealing with the development of the concept historically include (Malik, 1980) and (Kleiner, 1989). This issue has been influential on curriculum decisions about how functions should be taught; this was particularly significant during the so-called 'new math' era (see e.g., Eisenberg, 1991). This theme is rather incidental to this paper, but it is important to mention it because in the duality of covariation and correspondence, the correspondence is closely linked with the 'modern' definition of function. In contrast, though, in this paper we wish to stress differences between correspondences and functions.

We shall argue then that although a function can always be constructed to 'express' any given correspondence, the correspondence and the function are essentially different things. In particular the correspondence is always understood within the context of a task environment, so we avoid having to think about the mapping explicitly in the form of ordered pairs (that is a problematic structure for many students if they are exposed to it). The correspondence involves an assignation rule that maps one family of objects to another in a systematic way. The function is a framework that allows the expression of this assignation rule in terms of specific sets. As such, we regard the function as a formal counterpart of the correspondence that is taken as having an intuitive character.

In this paper we shall develop the issue of the previous paragraph, and shall discuss its educational significance. In particular, now we are in the position of comparing an intuitive construct with one that is more formal, a ripe situation for

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balancing flexible thought with tight argumentation. We shall illustrate this by describing a particular episode from a fieldwork that displays how the central idea opening up a general direction for solving a specific task comes in the form of a correspondence, but this correspondence is handled securely only when it is 'converted' into a function.

Correspondences, functions and assignation rules

Perhaps some channels of communicating mathematics, especially textbooks, would not explicitly discriminate between the terms 'function' and 'correspondence'. This is because they may not aim to provide means for overt discussion between the intuitive and formal levels of expression; evidently, though, the two words carry different connotations.

A correspondence marks the outcome of a mental activity that led to the identification of some association in a system. The 'human' component here should be stressed. We should also qualify what we mean by 'association' in this context. What governs the character of the association is a natural rule (i.e., the rule is determined by the perceived structure of the system) that provides a particular reason to make a mental linkage between any object of one kind with a (unique) one of another kind. We shall call such a rule an assignation rule. The assignation rule has the role of indicating how any particular relevant single object would be assigned or related to some other object. The associated correspondence expresses the 'objectification' of the process implied by the rule, in the sense that one can conceive the rule acting on all relevant objects simultaneously. We stress here that although there is a consciousness of a systematic pairing of objects, a correspondence in itself does not imply an analysis of the exact range, or extent, of the objects for which it makes sense to say that the rule applies. The assignation rule is the prime focus for the correspondence; a common occurrence, though, is that a student understands the rule but tries to apply it where this is not suitable. (The basic rule may be constrained by certain conditions due to the specific aims of the mathematics being done.)

In this paper we choose to consider the role of a function as a counterpart to a correspondence. (In more abstract situations, e.g., when a certain function is known to exist but is not explicitly constructed, there may be no sense of an accompanying correspondence.) The formal definition of a function is strictly given in terms of a (formal) relation, but what is usually used is the medium of the sets domain, co-domain and a means to associate every element of the domain with a (unique) element of the co-domain. A function differs from a correspondence in two ways. First, the function insists on the specification of two sets before any association is to be introduced. This contrasts with the focus that a correspondence has, where an assignation rule is comprehended first contextually, and only after can there be reflection as to the exact range of the objects involved. Second, although a function must express a means to make an assignation, this does not necessarily have to be explicit or to take the form of anything that would be recognized as a rule. How can

these differences be explained, and what possible cognitive problems might accrue for the student?

What is at the heart of this issue possibly is the following: the genesis of a correspondence is in the mental processing of an observation, whereas the role of a function is more to do with control. Let us expand on this. A correspondence usually results from the identification of an association discovered in terms of the system being studied. Quite likely this discovery will affect the student's cognitive view of the whole system. One thing that can happen is that too much attention is put on the newly realized correspondence, with the outcome that the original aims of the task become confused. Insisting on specifying explicit sets describing the objects to be related would help in controlling the situation. This forces a shift of attention from the correspondence itself (primarily thought about in terms of an assignation rule) to the delineation of the 'arena' for which the correspondence applies. Identifying this 'arena' should provide a balanced perspective about the correspondence within the whole system.

From the discussion above, it is clear from a cognitive point of view that a function formed to 'reflect' a given correspondence does not have the character of a replica in more mathematical terms, nor even of a mathematical model. The correspondence and the function have essential differences that hint that the latter is best thought of as a way of accommodating the former in a controlled mathematical environment. It may be difficult for students to appreciate the two different but related roles taken here. Indeed the educational literature on functions suggests that students do not make a clear distinction between a correspondence and its parallel function. A typical list of students beliefs on functions, as in (Vinner, 1983), hardly would refer to domains and co-domains. Instead it would mostly concern relationships or the usage of given rules, together with the idea of covariation, as well as identification with specialized types of representation such as algebraic formulae and graphs. Covariation provides an alternative intuitive way to process relationships, see the introduction. All of the other items tend to promote the idea of the agency of an assignation rule, and hence would suggest a mentality more allied to a correspondence rather than to a function (from our point of view). Further, certain student behavior, such as over assumption of the property 1:1 in relationships as mentioned in (Dubinsky & Harel, 1992), could be accounted for from the correspondence viewpoint. But what consequences might this over-riding dependence on the correspondence have in real terms?

If, as the evidence indicates, students do not appreciate the role of the definition of a function as an accommodating framework, according to our theoretical position they may well lack the control in refining correspondences. We shall raise a particular point concerning this situation. If a correspondence were realized within a system, would the significance of the correspondence in the general system deflect attention from the specific task aims? Were the respective function formulated, would this

circumstance be better controlled? In the next two sections, we will illustrate this issue by an episode extracted from some fieldwork.

The Fieldwork

The original purpose of the fieldwork was to illustrate a teaching sequence designed to make students become aware of a particular technique (but with the potential of being used for other techniques). The particular technique studied was the construction of a bijection in order to transfer questions about how many elements there are in one set to another. The main instrument in the teaching sequence is a semi-structured discussion between a small group of students with a teacher/researcher prompting its overall direction. The students try to answer some tasks all designed such that they are most conveniently solved via the technique. The prompting is done to help the students to achieve a solution consonant to the technique for each task, but this is not (necessarily) done with the students' being aware of the technique itself. After, further prompting is performed to make the students reflect about the solutions of the tasks and their commonality, in the hope that this would yield a conscious awareness of the technique.
In practice, what the first level of prompting involved was to 'nudge' the students' attention towards a particular relationship understood in terms of the task environment. The second level of prompting largely concerned influencing the students to try to express the relationship as an explicit function. Thus we have the situation where correspondences are first observed in context and then are accommodated as functions, just as in the main theoretical theme presented above. This explains why this fieldwork is pertinent.

As we shall only extract one particular episode from this fieldwork, more functional details are not given here. The study took place at an U.S.A. University in 2001 involving 4 sophomore students all planning to major in disciplines with high mathematical requirements. The discussion was audio and video taped.

The Episode

We shall present an episode extracted from the fieldwork described in the previous section. The participating students will be denoted as S1, S2, S3, S4 and the prompter as L. Material in parenthesis in the transcript is explanatory and not spoken. The task considered is:

A <u>(r,m)-tuple</u> is an ordered string of 0's and 1's where there are r 1's and (m – r) 0's. We denote the set of all (r,m)-tuples by $S_{r,m}$.

Form a bijection between $S_{r-1,m-1} \cup S_{r,m-1}$ and $S_{r,m}$. Explain why this implies: ${}_{r-1}C_{m-1} + {}_{r}C_{m-1} = {}_{r}C_{m}$

where ${}_{r}C_{m}$ signifies the number of ways of picking r things out of m.

The key in solving the problem is to recognize the natural assignation rule suggested by the action of suppressing the, say, last component of an element of $S_{r,m}$. More specifically, if the last component of an element of $S_{r,m}$ is 0, it is mapped to an element of $S_{r,m-1}$; if it is 1, it is mapped to an element of $S_{r-1,m-1}$. Realizing that the underlying function is a bijection, and that $| S_{r,m} | = {}_rC_m$, this yields the well known numerical identity between binomial coefficients without algebraic manipulation.

The students were not able to proceed on the task on their own. Prompting was done to draw the students' attention to the number of components of the elements in the given sets. This eventually led one student, S2, to state: "Couldn't you just like knock out the last...?" At this point we expected a clear expression of a correspondence to emerge; instead the discussion took another turn:

1. S3: The m term will either be a 1 or a zero. So if it's 1, send it to this set ($S_{r-1,m-1}$) and if it's a zero, send it to the other set ($S_{r,m-1}$).
2. L: Uh-huh. So…
3. S1: … Well, the only way that would work, mapping the ones with zero in the mth term to the first one ($S_{r,m-1}$)., and with 1 in the mth term to the second one ($S_{r-1,m-1}$), the only way that would work is if the two sets have an equal number of elements. That would be the only way that I can see that would be a bijection.
4. L: Can you explain yourself a little bit more on that?
5. S1: Well, the two possibilities are that it has zero in the last term or 1 in the last term. And there are equal number of these terms and these terms… There is an equal number of each… Okay, if there are m terms in this ($S_{r,m}$), there are m over 2 terms of this ($S_{r,m-1}$) and there are m over 2 terms of this ($S_{r-1,m-1}$) .
6. S3: They ($S_{r-1,m-1}$ and $S_{r,m-1}$) would have to have the same number of elements as half of that set ($S_{r,m}$) ?
7. S1: Well, I was assuming that since we said that a bijection existed, that we're trying to find a bijection, if a bijection did exist. Then this set would have to have the same number of elements as this set anyway. So… I took that on an assumption. But, yes, that's true.

   (Some discussion suppressed).
8. S3: I have something. In the case where the mth term is zero, we know there are r 1's. And if you take the m-1 elements and you could match them up to $S_{r,m-1}$, that would form half of the bijection. And in the bottom case… in the bottom case you know there are r 1's, and one of them is already the mth term so there are r-1 left, and you could map that to that ($S_{r-1,m-1}$) … and that would be a bijection.
9. L: Okay, so what we had before, I think we had the suggestion that the size of this set ($S_{r-1,m-1}$) and this set ($S_{r,m-1}$) was equal…

10. S1: I'm thinking that's probably not true, no. I was thinking that there were… Okay, no, that's not true at all. I take that back. I'm agreeing with what he's saying now, that that's not important. Okay, I understand.

The students are led to observe a particular action (of 'knocking out' the last component) that induces a way to identify any object of one kind (a tuple with m components) with one of another kind (a tuple with m - 1 components). This means the students have produced a natural assignation rule and thus a correspondence.

The task environment refers to certain sets (of tuples), whose appearance (we supposed) would help the students to make a smooth transfer from the correspondence to a suitable function. As we see from the protocol above, this transfer in fact took some time to be effected. In 1, we see that the correspondence was mentally reprocessed in a coarser way to how it was first conceived. Instead of associating a tuple T (in $S_{r,m}$) with a particular tuple with m-1 components, the rule is re-read only to register to which out of the two sets, $S_{r-1,m-1}$ or $S_{r,m-1}$, T will be 'designated' to. In this way the focus is turned away from the natural 1:1 matching that the correspondence suggests. It seems that this image dominates until we get to 8. The correspondence is being used to gain a particular perspective on the general system, but it is not being developed as a function. From this situation and as the task environment explicitly asks for a bijection, the students actually look elsewhere to find a bijective function. One student claims that the sets $S_{r-1,m-1}$ and $S_{r,m-1}$ have the same number of elements in 3, a proposition that is not (in general) correct. There seemed to be two reasons why the student makes this proposition. First in 5 he expressed the (false) belief that the number of elements of $S_{r,m}$ 'ending' with 0 equals the number of elements 'ending' with 1. (We conjecture that the source of this belief is based on an under criticized sense of symmetry.) The understood designation of sending elements of $S_{r,m}$ into either the set $S_{r-1,m-1}$ or the set $S_{r,m-1}$ then leads him to his claim. Second, in 3 and 7 it comes clear that he is also influenced by the appearance of the word 'bijection' in the question; he 'knows' that he must fit one somehow in the system and the only way he can 'see' one is through the supposition that $|S_{r-1,m-1}| = |S_{r,m-1}|$ that would ensure that a bijection exists between these two sets. (Notice in taking this stance means that no bijection has been explicitly constructed.) Only in 8, that occurred a significant time after the start of this discussion (some material has been omitted in the protocol), did another student identify a bijection with the original correspondence. This led the student S1 to immediately renounce his claim in 10. Notice how this student was not interested in explaining explicitly <u>why</u> his claim was false, but he dismissed it as if there was no longer any reason to believe that it should be true.

We find this episode interesting because it illustrates how a correspondence observed in a system may not be readily converted into a function, but can be an influence to throw a new intuitive perspective of the system. However doing this led to an unhelpful line of thought. Only when finally the correspondence was fully integrated with the sets being talked about (so that it could be recognized as a

function) did this confusion dissipate. Hence the forming of the function acted as a device of control in the soundness of argument made about the correspondence.

Concluding Remarks

This paper has discussed the process of forming functions from the basis of observing correspondences drawn from context. In this respect we make a differentiation between the character of a correspondence and that of a function. We believe that this perspective is not well represented in the extant literature on functions. We agree with the argument given in the review article (Thompson, 1994) that the current trend of concentrating on the so-called representations of functions (usually graphs, algebraic expressions, tables) and their co-ordination is being over emphasized. This tendency reflects the current teaching practices that has severely restricted the students' image of what functions are in general, as this image is dominated by certain paradigms, see (Bakar & Tall,1991). Further, the representation perspective tends to neglect the question how, and indeed why, functions come to be constructed. There are studies that go some distance in this direction, where within the task environment a certain relationship is pointed out, and the task itself is to express it by a suitable real function (sometimes where only qualitative description in terms of properties is feasible, see e.g., Monk, 1992 Thompson, 1994). What often seems to occur in such studies is that students lose control concerning what families of objects should accommodate the intuitive sense of the relationship. In this paper, we go a little further in that even the obtaining of the relationship is part of the solving procedure. It is perhaps in the latter circumstance that the drawing of a distinction between a correspondence and a function is at its most compelling. The correspondence is how a suitable relation observed in a system is first thought of, and the allied function has the role to ensure that the subsequent mental argumentation of the correspondence is grounded on an explicit mathematical framework that should remove vagueness and arbitrary interpretations. A crucial part of this is that the function accommodates the assignation rule understood for the correspondence in an unequivocal way. This issue was well illustrated by the episode described in this paper. We believe that the perspective that we have laid down on correspondences, functions and assignation rules should provide not only a good way to explain students' behavior whilst constructing functions, but should be taken in account in how functions are taught. We plan to expand on these themes in subsequent papers.

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UNJUSTIFIED ASSUMPTIONS BASED ON DIAGRAMS IN GEOMETRY

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We investigated unjustified assumptions made by students when proving geometric statements. Geometric statements can be presented with a diagram or without. Diagrams can be accurate or sketchy. Unjustified assumptions may originate in an accompanying diagram. We thus asked whether the way in which a statement is presented has an effect on unjustified assumptions. We also attempted to find out what motivates students to make unjustified assumptions. Data were collected by means of written questionnaires and individual interviews. The main findings were that among all incorrect answers, 72% were based on unjustified assumptions, and that students make unjustified assumptions with good reasons such as in order to remove obstacles.

THEORETICAL BACKGROUND

Fischbein (1993) introduced the term "figural concepts" to stress the double nature of geometric figures: conceptual and figural. The conceptual nature includes characteristics such as completeness, abstraction and generalization while the figural nature includes characteristics such as color, size and shape. The conceptual and figural characteristics used when proving depend both, on the conceptual system that includes abstract ideas and concepts and on the figural system that includes mental representations and images. For example, when attempting to make two triangles overlap, concepts like angle, side and triangle are needed as well as figural information like suitable angles and sides (Tall & Vinner, 1981). In every process like this, there is a tension between the conceptual system and the figural system and many of the difficulties in geometry can be interpreted due to this tension. Definitions, concepts and theorems impose characteristics on geometrical objects. However, these definitions and concepts are not always clear to students and often they are forgotten. As a result, the figural component tends to free itself from the formal control and to act independently (Fischbein, 1993; Mariotti, 1997). As a result, diagrams in geometry can be obstacles when proving geometrical statements. These obstacles are divided into three types:

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Particularity of Diagrams: Most diagrams in high school geometry are intended as models. They are meant to be understood as representing a class of objects and contain the essence of the situation. Nevertheless, every diagram has characteristics that are individual and not representative of the class. For example, a specific acute triangle ABC which is meant to represent all triangles is by no means a universally valid representation since it does not depict obtuse angles. This obstacle causes students to be trapped by the one case concreteness of an image or diagram which may contain irrelevant details or may even introduce false data (Yerushalmy & Chazan, 1990).

Prototypical Diagrams as Models: If students link a definition to a standard, prototypical diagram, the particularity of the diagram can lead to another obstacle: a prototypical image may induce inflexible thinking thus preventing the recognition of a concept in a non-standard diagram. Students' definitions may include irrelevant characteristics of the standard diagram, causing difficulties in creating or interpreting diagrams (Yerushalmy & Chazan, 1990). For example, among rectangular triangles, the one with the perpendicular sides in vertical and horizontal position is prototypical and students as well as teachers have great difficulty in identifying other positive examples (Hershkowitz, 1989).

Inability to "See" a Diagram in Different Ways: Psychologists often test spatial ability by means of embedded figures tasks, in which a simple figure must be identified in a more complex figure. Yerushalmy & Chazan (1990) consider this sort of reorganization to be a central aspect of mathematical creativity. In geometry, there are situations, where students are asked to do this sort of reorganization. However, the ability to attend selectively to parts and whole does not come easily for many students. According to Hoffer's (1981) formulation, the van Hiele stages suggest that at level 1 (recognition) the student recognizes a shape as a whole. It is only at level 2 (analysis) that the student can focus on parts of a diagram and analyze properties of figures. For example, students may not be able to see AD as a side of triangles ABD and ACD because it is seen only as the altitude of triangle ABC (Fig.1) (Hoz, 1981).

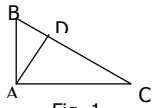


Fig. 1

These three obstacles may lead students to make unjustified assumptions, i.e. to assume properties that are not given and are not essential for proving the statement at hand. Geometric statements can be presented with a diagram or without. Diagrams can be accurate or sketchy. Unjustified assumptions may originate in an accompanying diagram.

This led to the following research questions:

1a) Do students make unjustified assumptions, when proving geometric claims?

1b) Do students make more unjustified assumptions when the statement is given with an accompanying diagram or when it is given without diagram?

1c) Do students make more unjustified assumptions when the diagram accompanying the statement is accurate or when it is sketchy?

2) What motivates students to make unjustified assumptions?

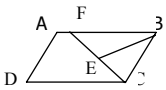
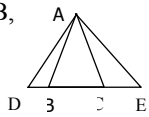
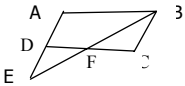
METHOD

In order to investigate these research questions, data were collected by means of written questionnaires and by means of individual interviews. Questionnaires were administered to 92 students from four classes in three different schools in Israel who were enrolled in a full-year 10th grade geometry course. Seventeen of these students were subsequently interviewed. Interviews were audio-recorded and transcribed.

Research Instruments

Questionnaire: Three questionnaire versions were used. All versions included the same three statements and proof tasks: two about parallelograms and one about an isosceles triangle. However, the statements were presented differently in the different versions. The tasks were all within the field of experience of the students, and of a level they could be expected to prove in class or in an examination (Table 1).

Table 1: The questionnaire tasks with the sketchy diagram

Statement	*Task*	*Goal*
Given: ABCD parallelogram. BE, CF bisect angles B, C. 	Prove: FE=EC	To investigate whether students make unjustified assumptions, such as FB=CB or $BE \perp FC$ with an accurate and a sketchy diagram and without diagram.
Given: AC=AB, <DAB=<CAE 	Prove: DB=CE	To investigate whether students make unjustified assumptions, such as AD=AE, <D=<E with an accurate and a sketchy diagram and without diagram
Given: ABCD parallelogram. Point F is the middle of DC 	Prove: AD=DE	To investigate whether students make unjustified assumptions, such as: BF=FE with an accurate and a sketchy diagram and without diagram

Each statement was presented with an accurate diagram in one version, with a sketchy diagram in another version and without diagram in the third version. Each version included one statement with an accurate diagram, one statement with a sketchy diagram and one statement without diagram. The order of presentation of the statements was the same in each version (Table 2).

Table 2: The three versions of the questionnaire

Task	*Version A*	*Version B*	*Version C*
1	accurate diagram	sketchy diagram	without diagram
2	without diagram	accurate diagram	sketchy diagram
3	sketchy diagram	without diagram	accurate diagram

Each student was given one version of the questionnaire at random. Writing the questionnaire in three versions was intended to

a) eliminate the influence of any particular task or of any particular representation,

b) separate between the three ways of presentation (accurate diagram, sketchy diagram and without diagram), in order to get a broad picture about the influence of each way on students' proving and making unjustified assumptions.

Interviews: Seventeen individual interviews were conducted in order to explore what motivates students to make unjustified assumptions in tasks with an accurate diagram, in tasks with a sketchy diagram and in tasks without diagram. The interviewees were selected according to three criteria:

- between three to five students were selected from each class, in order to represent
- each one of the classes appropriately, five or six students were selected for each version of the questionnaire, in order to get an appropriate picture of each one of the three presentations,
- the students were selected according to their performance in the questionnaire.

FINDINGS AND DISCUSSION

Table 3 presents the results of the three tasks according to the following categories: correct answers, incorrect answers and no answer. The category "correct answers" includes only answers with satisfactory proofs of the statement; all answers with unjustified assumptions were counted incorrect. The numbers in brackets in the cells of the incorrect answers refer to unjustified assumptions.

Table 3: Distribution of correct and incorrect answers (N=92)

Task	*Correct answers*	*Incorrect answers*	*(unjustified) (assumptions)*	*No answer*	*Total*
1	60	27	**(22)**	5	92
2	75	9	**(5)**	8	92
3	73	14	**(9)**	5	92
Total	208	50	**(36)**	18	276

These results show that the majority of students responded correctly to the tasks, while some did not respond at all. But our focus is on the incorrect responses. The important finding is that among the 50 incorrect responses, 36 (72%) were based on unjustified assumptions. Furthermore, in each task, more than 50% of the incorrect responses were based on unjustified assumptions. These unjustified assumptions appeared in 36 responses and were due to 33 students (36% of the population). Thirty students made unjustified assumptions in one task and three in two tasks. This answers research question 1a.

In order to answer research question 1b, we compare the influence of the two presentations, with diagram or without diagram, on unjustified assumptions. Similarly, in order to answer research question 1c, we compare the influence of the two kinds of diagram, accurate or sketchy, on unjustified assumptions. Table 4 presents the results according to the three kinds of presentation.

Table 4: Unjustified assumptions according to presentation (N=36)

Task	*Accurate diagram*	*Sketchy diagram*	*Without diagram*	*Unjustified assumptions (Total)*
1	6	5	11	**22**
2	0	2	3	**5**
3	3	2	4	**9**
Total	9	9	18	**36**

Table 4 shows that the number of unjustified assumptions in tasks without diagrams was twice as large as in tasks with diagrams and that no difference was found between the number of unjustified assumptions in tasks with accurate diagrams and in tasks with sketchy diagrams. This answers research questions 1b and 1c.

Table 5: Number of students making main unjustified assumptions

Task	*assumption*	*This assumption*	*Unjustified assumptions (Total)*
1	Drawing the line FG, where G is the intersection point between BE and CD, and assuming that it parallels to BC	11	22
2	AD=AE <ADE=<AED	2 2	5
3	EF=FB	3	9

Table 5 presents in the first column the most frequent unjustified assumptions, in the second column the number of students who made these assumptions and in the third column the total number of students who made unjustified assumptions. For each unjustified assumption, a detailed analysis was carried out according to the following aspects:
a) Motives for making the unjustified assumption,
b) Using the unjustified assumption for proving the relevant statement,
c) Using the unjustified assumption with a backward or forward view,
d) Awareness of using the unjustified assumption,
e) The effect of the presentation on unjustified assumptions.

In this paper, we present this analysis for only one unjustified assumption, namely the one in Table 5, which was made in task 1. This assumption was the most frequent unjustified assumption made in this study. However, we then present an overview of the results of the analysis of all unjustified assumptions.

As mentioned in Table 5, the main unjustified assumption in task 1 was drawing the line FG, where G is the intersection point between BE and CD, and assuming that this line was parallel to BC. The most common motive for this assumption was to create a parallelogram FBCG (aspect a). Since FBCG is a parallelogram, its diagonals bisect each other and it was therefore easy to complete the proof (aspect b). Almost half of the students who made this assumption, did it with a backward view: looking at the givens, they already had a plan in their mind leading them to the end of the proof. The remaining students did it with a forward view: they considered how to use the givens in order to solve the task (aspect c). Most of the students who made this unjustified assumption were not aware of their mistake.

During the interviews, they were very sure about their answers and did not hesitate at all (aspect d). This assumption was made about equally frequently in all three presentations. Thus, in this case, the presentation had no effect on the unjustified assumption (aspect e).

Discussion according to aspects a-e

a) Motives for unjustified assumption: Two main motives were found: a mathematical motive and a purely visual motive. According to the mathematical motive, students made unjustified assumptions in order to reach a specific stage in the proof, which led easily to the given statement. According to the purely visual motive, the way the diagram looked strengthened the students' feeling and intuition that the stage they wanted to reach (according to the mathematical motive) was correct, because they could see it in the diagram.

b) Using the unjustified assumption for proving the given statement: Students made unjustified assumptions instead of propositions they wanted to reach but didn't know how, or instead of propositions that they believed were correct.

c) Using the unjustified assumption in a backward or forward direction: The students who made unjustified assumptions, did it with one of two views: backward or forward. Those who did it with a backward view, thought of the stages they had to reach, in order to prove the given statement (from end to beginning). Those who did it with a forward view, thought how to use the givens and the relevant theorems and propositions in order to reach the end of the proof (from beginning to end).

d) Awareness of using the unjustified assumption: Fourteen out of the seventeen interviewees were not aware at all of their unjustified assumptions, neither when answering the questionnaire, nor during the interview. Two others became aware during the interview. Only one student mentioned that she was aware of the unjustified assumption while solving the task. She explained that she felt the assumption was correct, but she did not know how to reach it correctly.

e) The effect of the presentation on unjustified assumptions: In almost each task, there was a purely visual motive in aspect a. This motive demonstrates that the diagram affected the students' way of thinking and making unjustified assumptions. There were also cases where students, themselves, built diagrams, in which the unjustified assumptions were shown. Therefore, they used their diagrams as evidence why these assumptions were correct.

CONCLUSIONS

- About four fifths of all answers were correct and only one fifth was incorrect. However, among the incorrect answers, 72% were based on unjustified assumptions.
- The number of unjustified assumptions in tasks without diagrams was twice as large as the number of unjustified assumptions in tasks with diagrams.
- No difference was found between the number of unjustified assumptions in tasks with accurate diagrams and the number of unjustified assumptions in tasks with sketchy diagrams.
- Unjustified assumptions were made with the purpose of reaching a critical step in the proof.
- Unjustified assumptions facilitated dealing with the tasks, removed obstacles and led immediately to the goal.
- Unjustified assumptions were made when students believed they were correct.
- Unjustified assumptions were made when students were stuck.
- In most cases, students made unjustified assumptions without being aware.

In order to prevent the development of misconceptions regarding this phenomenon, teachers should be equipped with appropriate tools for working with their students and have to suggest a variety of problems given in the three ways of presentation.

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THE IMPACT OF INDIVIDUAL CURRICULA ON TEACHING STOCHASTICS

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This report focuses on teachers' individual curricula. An individual curriculum includes contents and reasoning and can be structured in a quasi-logical system of goals and methods, which is the result of teachers' planning of mathematics instruction. There is consent that the planning of individual curricula or the instructional practice is a form of social action. While action is an inner and subjective process, which is dependent on situations and individuals' interpretation of a situation, here, the approach of research is qualitative and interpretative. So, individual curricula are re-constructed from interviews held with eight secondary teachers. One of the eight cases, the one of Mr. A, is lined out here.

THEORETICAL FRAMEWORK

Curricula change. Here, curriculum means the system of subjects of instruction and reasoning of this system and issues that are directly related. While the contents of teaching as well as the main goals of teaching different mathematical disciplines are similar or identical, the importance of specific contents and the system of reasoning depend on the development of theories of teaching (mathematics). As the development of new curricula follows social or political requirements or new didactical knowledge, it must be the main goal of professionals responsible for developing curricula to realize new ways of learning and teaching in schools. In every learning theory, the key persons to apply new curricula to enable students to acquire (subjective) knowledge are teachers (see Fernandes 1995 and Wilson/Cooney 2002). Only they choose the subjects of instruction and only they define their goals for teaching mathematics.

There are – especially in Germany – two ways of realizing new curricula. In the revolutionary way, curricula are published by state governments and have to be installed in schools. But, especially in Germany, research shows that this way does not work. Governmental curricula and even didactical proposals for modified curricula are obviously realized seldom in the daily instruction of mathematics (see Vollstädt et al. 1998). This is (in Germany) especially the case for stochastics. Over 40 years, it is a didactical demand to teach stochastics or to teach more stochastics, but stochastics are hardly present in today's mathematics instruction. So what may be the key of changing mathematics and stochastics instructional practice?

In the evolutionary way, teachers integrate step by step didactical proposes in their individual curricula in an active and self-determined way. Here, it is one main hypothesis that understanding teachers' individual curricula is mandatory to grasp their instructional practice and to be able to change this practice (see also Pehkonen/Törner 1993). So teachers' individual curriculum, their subjective knowledge and concep-

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tions about mathematics and about learning and teaching mathematics is the focus of this line of research.

Governmental and didactical curricula consist of contents and their reasoning. Reasoning means that methods and goals are connected in form of if-then-sentences (see König 1975). One example:

Students have to learn data-analysis. (goal)		
If students learn data-analysis	*then*	they will become an individual with the ability to criticize. (method)
		Students have to become an individual with the ability to criticize. (goal)

Figure 1: Goals and methods

Here, it is another main hypothesis that individual curricula are constructed in the same way. The system of reasoning called goal-method-argumentation shall broaden and deepen the results of research on beliefs postulating types of teachers and their individual curricula (see Thompson 1984, Thompson 1992 and Leder, Pehkonen and Törner 2002).

There is consent that the planning of individual curricula or the instructional practice is a form of social action. While action – in sociological as in psychological definition – is an inner and subjective process, which is dependent on situations and individuals' interpretation of a situation (see Wilson 1973), here, the approach of research is qualitative and interpretative and uses the well elaborated psychological approach of subjective theories (see "Forschungsprogramm Subjektive Theorien", Groeben et al. 1988). This approach newly developed in contrast to the behavioristic way by understanding people's acting. It proposes the epistemological modelling of human theories of actions, which are parallel to researchers' theories. The approach is oriented on Kelly's (1955) report about the "man as scientist" and focuses on people's subjective knowledge structured in quasi-logical systems of concepts. Groeben et al. (1988) postulate the goal-method-argumentation to be one of these quasi-logical systems of concepts.

So, the question of research focused on in this report is:

What are the contents and goals of teachers' individual curricula of stochastics?

What are the goal-method-argumentations of teachers' individual curricula and how they structure teachers' instructional goals?

METHODOLOGY

The approach of understanding action as an inner process depending on situations determines an inquiry in form of case studies (see Stage 2000). The definition of the cases is according to the theoretical sampling (see Glaser/Strauss 1967). Here, the cases were eight teachers grade 7 to 13 of secondary schools (A-level)[1] in Northern Germany. The reconstruction of the individual (stochastic) curricula is based on interviews designed in form of the problem-oriented interview (see Witzel 1982). The topics of these interviews emerged form the analysis of didactical curricula.

The interviews were taped and transcribed. The interpretation was done according to the main method of humanities, the hermeneutics (see Gadamer, H. G. 1986 and Danner, H. 1998). Firstly, subjective concepts or goals of instruction and their definitions were reconstructed. The second step of reconstruction included the reduction of subjective concepts to main concepts and the construction of a system of goals and methods and their relations in form of if-then-sentences (goal-method-argumentation). This step determined the differentiation between five aspects of an individual curriculum: The contents of instruction, the goals of stochastics and mathematics instruction, teachers' knowledge about how students view the usefulness of mathematics and finally, teachers' knowledge about teaching mathematics successfully (see to the latter two also Brown 1995). The identification of patterns of argumentations and the definitions of types across individual curricula is not subject of this report.

The following discussion focuses on the results and their interpretation. The process of interpretation of the interviews and the reconstruction is not lined out. While primarily one case will be discussed, some results of the other cases are used to complete the case description.

THE CASE OF MR. A

A is a teacher at a gymnasium in a little town in Northern Germany. As only three of twelve of A's colleagues teach stochastics, A has to start with elementary fundamentals of stochastics in grade 10 and grade 13, where he teaches stochastics.

The curriculum concerning the subjects of instruction is shown below (see figure 1).

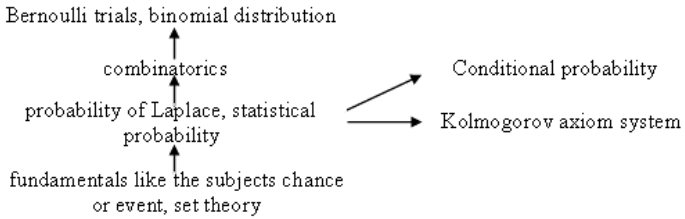

Figure 2: Subjects of instruction

There are four characteristic aspects to this curriculum. Firstly, A defines school-stochastics as theory of probability. Elements of data-analysis such as descriptive statistics or statistical inference are missing. Furthermore, the curriculum is limited. Up from elementary fundamentals it will end with, also in the highest grade, an introduction to binomial distributions. Some of these contents (see figure 1) are part of all individual curricula analysed and are summed up to the category of a classic block of theory of probability. This means the sequence: fundamentals (for example chance or event), probability, combinatorics, Bernoulli-experiment and binomial distributions.

The interpretation of the central idea of probability (the statistical or subjective interpretation or the interpretation of probability defined by Laplace or described by the axioms) has evolved to be a main criterion of the curriculum's analysis. There is one type of teacher like A, who anchors her or his curriculum in Laplace's interpretation of probability and limits the curriculum as shown above. For example, another type of teacher focuses on the statistical interpretation of probability. Her or his individual curriculum includes data-analysis and especially statistical inference (the differentiation is independent of the time-span teachers use for teaching stochastics). Instead of teaching statistics, A also covers subjects like Kolmogorov's system of axiom or the conditional probability, which are not necessary for the main curriculum. Especially teaching axioms fulfils A's idea of gymnasium's instruction of mathematics. So, on the one side A's curriculum is limited, but on the other side it is extensive within its limits.

These characteristics are integrated in the argumentation on the goals of teaching stochastics (figure 2 and also the following figures show a strongly condensed version of the goal-method-argumentations).

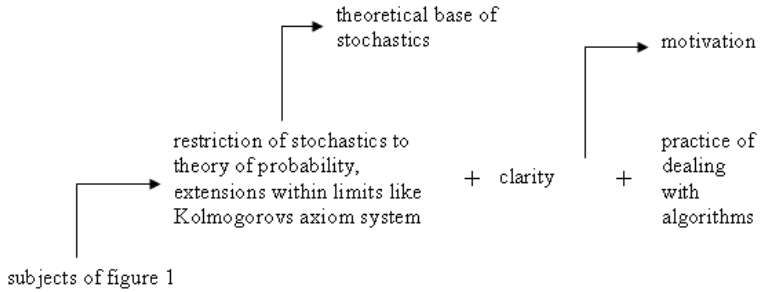

Figure 3: Goals of stochastics curriculum

The link between contents and goals of instruction is in every case one goal based on the thesis that the contents of instruction are the result of the teacher's conscious election. In the first level of goals there are some, which are concerning contents of instruction and are described above. Other goals include reasoning of instructional practice like using clear concepts in instruction, which are not based on special subjects of instruction and are meant to increase motivation. Here, motivation means students enjoyed doing mathematics and so efficient instruction is possible. Finally,

there is the goal, especially for weak students, to acquire the ability to use mathematical algorithms.

The subject-oriented goals are the main goals in A's goal-method-argumentation. So the main goal of stochastics is to build up a theoretical base of stochastics. This base does not enable students to solve real stochastic problems. It is only a base, which may be extended after leaving school. Beside stochastics algorithms it is the formal system of stochastics, axioms, definitions, theorems and proofs, which characterizes the theoretical base.

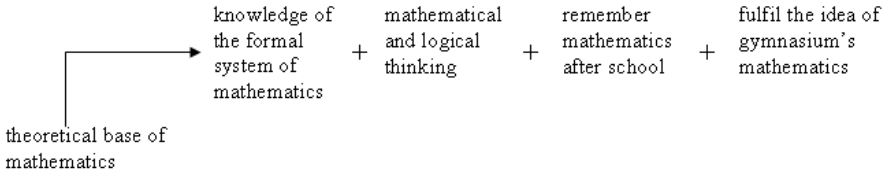

Figure 4: Goals of mathematics curriculum

As the theoretical base is the core of A's individual stochastics curriculum, it is the core of the mathematics curriculum, too. The knowledge of the formal and deductive system of mathematics according to school-mathematics and the ability of dealing with special mathematical algorithms is the prerequisite to achieve the highest goals of school-mathematics:

the knowledge of the formal and deductive system of mathematics;

the ability of mathematical and logical thinking. This means the ability to make deductive conclusions and to think of assumptions and conclusions;

the latter goals should be achieved in gymnasium's mathematics;

at last, it is A's opinion that only a solid theoretical base enables students to remember the mathematical subjects, methods, relations after their school-career.

For A, school-mathematics in general does not enable students to deal with real problems. While A's main goal is to convey the formal system of mathematics other teachers define goals concerning problem-solving or dealing with real problems.

In the context of another goal-method-argumentation, A's subjective knowledge about how students view the usefulness of mathematics is anchored in a goal discussed above, the ability of dealing with mathematical algorithms. This argumentation (see figure 4) is a pragmatic one. So students need this ability to manage school examinations. If they are successful, they obtain the permission to attend a university. If students view themselves as prepared for life they are satisfied with school, which is the highest goal in this argumentation.

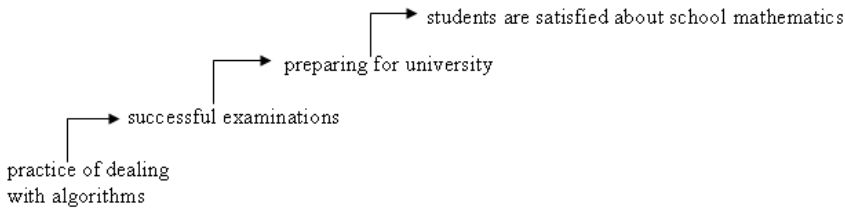

Figure 5: Students view on the usefulness of school-mathematics

In other cases, this argumentation does not only apply to prepare students for university or profession but also to prepare for life in terms of the ability of criticism.

A's last goal-method-argumentation (see figure 5) concerns his subjective knowledge about the efficiency of instructional practice. As sufficient exercise in dealing with mathematical algorithms results in success for most students, the clarity of instruction and, at last, an atmosphere of respect and understanding, however, makes the learning and teaching of mathematics possible. Firstly, it leads students to motivation. A's definition of three classes of efficient instructional practice is striking. Firstly, efficient instructional practice only means that students work and learn mathematical content. If this content has real applications or opens a deep insight into the formal system of mathematics, then A defined this as meaningful instruction. The third class is the combination of the latter two classes. A termed this worth-while instruction and stated that it is seldom realized.

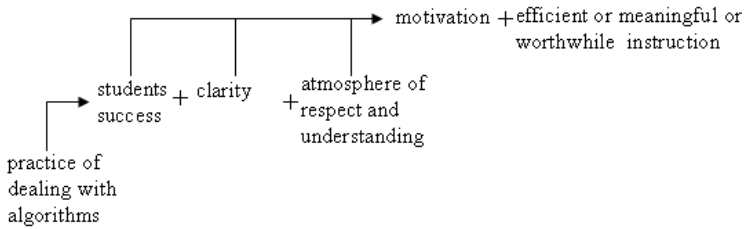

Figure 6: Efficiency of the instructional practice

DISCUSSION

A's individual curriculum – and also the other individual curricula – with special regards to stochastics consists of the discussed five aspects. The base of all goal-method-argumentations is the system of the contents of instruction. It is impossible to understand one's goals of stochastics or mathematics without knowing this base. The other two argumentations concerning content-oriented goals of stochastics and mathematics are well matched in the case of A. The other argumentations concerning students' use of mathematics and the functionality of teachers' instructional practice

are separate from special mathematical subjects. Finally, the following theses are based on this case description:

The five aspects of an individual curriculum discussed here are the main aspects of every individual curriculum.

In the eight individual curricula analysed there were no subjective concepts (or goals) that could not be ascribed to one of the five aspects. Also these five aspects are similar to theoretical differentiations of aspects of school-mathematics (see for example Thompson 1992).

Only the knowledge of all aspects leads to a real understanding of an individual curriculum.

One example: There is an open conflict in A's curriculum concerning the teaching of the formal system of mathematics or the algorithms as a toolbox. The main goal of stochastics and mathematics seems to be teaching the formal and deductive system. But the other argumentations show that for students' success and for efficient instruction it is necessary to reduce formalism and extend the exercise of algorithms. Furthermore, without knowing his goals oriented at the formal system it is impossible to understand some of A's contents of instruction like his teaching of the axioms or the conditional probability. So only the comprehensive analysis opens a real and deep understanding of A's individual curriculum.

Without teachers it is impossible to implement new ideas or developments of didactical curricula into schools instructional practice.

This hypothesis based on theoretical considerations and empirical results lined out in the discussion of the theoretical framework.

Without understanding teachers' individual curricula it is impossible to change these curricula.

One example: One new development of didactical curricula of stochastics is the extension to data-analysis. This could prove to be difficult for A, since there seems to be no anchor for integrating data-analysis into A's individual curriculum.

While teachers have to consider students' individual knowledge, also didactical curricula have to consider teachers' subjective knowledge and their individual curricula.

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[1] A-level means the German gymnasium. In Germany, there are generally three sorts of schools, the Hauptschule, the Realschule and the Gymnasium. The Gymnasium is a school with students from grade 7 to grade 13. If students manage the second level (grade 11 to grade 13), they are entitled to attend a university.

THE FUNCTIONS OF PICTURES IN PROBLEM SOLVING

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In the present study, we assert that pictures serve four functions in problem solving: decorative, representational, organizational and informational. We, therefore, investigate the effects of pictures based on their functions in mathematical problem solving (MPS), by high achievement students of Grade 6 in Cyprus, in a communication setting. A number of tasks were developed and techniques of observation and interviews were conducted for gathering qualitative data from eight students. All kinds of pictures, except the decorative one, were found to be conducive to MPS and the communication process. Findings also suggest that the use of pictures in successful MPS depends on the relationship between the picture and the task (function of picture), and on students' mental abilities.

THEORETICAL BACKGROUND

Bruner (1961) supports that learning proceeds through three levels: the enactive, the iconic and the symbolic. In other words, pictures function as a mediator between the practical and the theoretical formal level of understanding. In the field of mathematics learning and instruction, pictures play an important role as an aid for supporting reflection and as a means in communicating mathematical ideas. Many researchers consider imagistic representations as a fundamental cognitive system for mathematical learning and problem solving (DeWindt-King, & Goldin, 2003), while expert mathematicians as well as mathematics students perceive visual representations as a useful tool in MPS and frequently attempt to use them (Stylianou, 2001).

The process of visualization is considered to be indispensable in mathematics learning and more specifically in MPS. The important role of visualization in MPS is stressed by the findings of recent studies, which consider the ability to form images of mathematical relationships as a necessary presupposition for effective MPS (Brown, & Wheatly, 1997). In the context of MPS, visualization refers to the understanding of the problem with the construction and/or use of a diagram or a picture to help obtain a solution (Bishop, 1989). Researchers argue that the solution of a problem may be accomplished by using either visual representations, or analytic thought processes, or both. The analytic method involves cognitive handling of objects and procedures with or without the use of symbols (Zazkis, Dubinsky, & Dautermann, 1996).

The views supporting the use of visual aids or pictorial representations in mathematics do not diminish the importance of the verbal code of communication in the context of learning tasks or the effect of language on the formulation of thinking, communication and "doing mathematics" (Kieran, 2001). Language and pictures are

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considered as two distinct kinds of representation and communication of mathematical ideas, with fundamental differences in regard with their informational content, structure and usability (Schnotz, 2002), which complement each other.

Thomas, & Yoon Hong (2001) suggest that students can interact with a representation, including a pictorial one, in two ways, by observing it or by acting on it. In the present study, we consider that the active use of pictures in the process of MPS may lead students to internal conflicts, which can be meaningful and beneficial in the solution effort. These conflicts appear when the text of the mathematical problem leads to a different solution procedure than the one derived from the use of the picture associated with the problem. However, the use of pictures, images or diagrams may have negative effects due to the obstacles they can create (Bishop, 1989). These kinds of difficulties may be caused by factors, such as the selection and highlighting of some aspects of the picture at the expense of others, the emphasis on irrelevant details and the inappropriateness of spatial arrangement, which may give rise to misunderstanding (Colin, Chauvet, & Viennot, 2002).

Based on Carney and Levin's (2002) proposed functions that pictures serve in a text, this study used a similar categorization of pictures in order to examine the role of each type of pictures in students' performance, in MPS. This study proposes four functions (categories) of pictures in MPS: (a) decorative, (b) representational, (c) organizational, and (d) informational. Decorative pictures do not give any actual information concerning the solution of the problem. Representational pictures represent the whole or a part of the content of the problem, while organizational pictures provide directions for drawing or written work that support the solution procedure. Finally, informational pictures provide information that is essential for the solution of the problem; in other words, the problem is based on the picture.

What is new in this study is the investigation of the relationship between different types of pictorial representations and MPS. Specifically, the purpose of the study was to explore the role of pictures based on their function, in MPS by students of Grade 6, in the context of an experimental model of communication. Three research questions were accordingly formulated: First, which is the effect of each category of pictures on students' MPS performance? Second, which strategies do students use to solve a problem accompanied by each category of pictures? Third, which is the effect of each category of pictures on the communication between students during MPS?

METHOD

The study was qualitative in nature and employed an interaction process, which included both oral and written responses to tasks, based on the communication model introduced by Weber-Kubler (1981). Eight students of Grade 6, presenting high performance in mathematics, participated in the study. Four problems corresponding to the four functions of pictures, mentioned above, were used. The subjects were separated in couples and the administration procedure was repeated four times, according to the communication model shown in Figure 1 below.

The communication model allowed for the observation of the way students work and interact, and helped us determine whether students solve and communicate each problem with or without the use of the picture accompanying it. Specifically, the researcher initially described verbally the text of the problem and the relevant picture to Student 1, who tried to solve the problem. When he finished, he was given the written form of the problem along with the relevant picture and allowed to modify his previous solution or solve the problem again from the beginning, if necessary. Next to this, Student 1 described the particular problem to Student 2, playing the role of the researcher and the previous procedure was repeated, as shown in Figure 1.

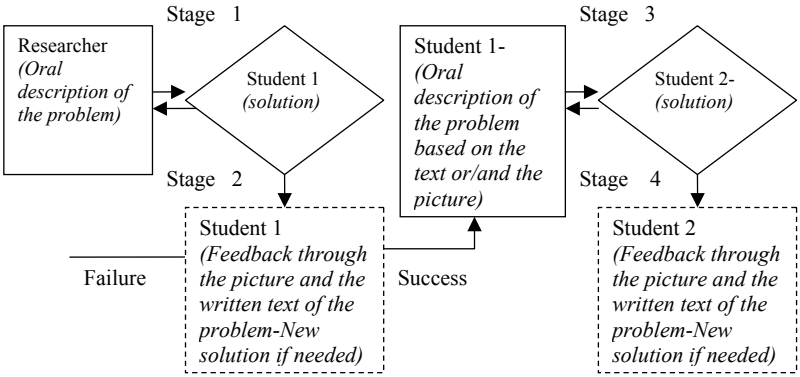

Figure 1: The procedure of the communication model

During the communication process, students could use paper and pencil in order to present their solution procedures, in written form, before and after the use of the picture. They were asked to explain and justify their solution strategies and determine whether the picture was useful for solving the problem, and if so, in what way. Each experiment lasted nearly 80 minutes for each couple and was recorded.

The tasks, which were based on mathematical problems used in recent studies or publications, are the following:

1. *Mrs Brown put her students into groups of 5, with 3 girls in each group. If Mrs Brown has 25 children in her class, how many boys and how many girls does she have?* (Misailidou, 2003)

The picture, which accompanied the problem, was decorative, and just represented a boy and a girl.

2. *Find the weights of the three items: a tetrahedron, a sphere and a cube, based on the information shown in the picture.* (Olson, 1998)

The picture (informational) consisted of all the information of the problem. It represented different combinations of the items on three scales: on the first scale there

were a tetrahedron, a cube and a sphere and the indication was 23 kg, on the second scale there were two cubes and two spheres and the indication was 22 kg, and on the third scale there were two tetrahedrons and a sphere and the indication was 28 kg.

3. *A man planted a tree at each of the two ends of a straight path. He then planted a tree every 2 m along the path. The length of the path is 10 m. How many trees were planted at the path altogether?* (Booth, & Thomas, 2000)

The picture of this problem had a representational role. It represented a path with two trees, one at each end of the path.

4. *Last week 10 campers camped at the mountain. Each day 8 loaves of bread were available for them to eat. This week 15 campers camped at the mountain. How many loaves are there available for them for the day?* (Misailidou, 2003)

The picture had an organizational role for the problem. A horizontal line separated the data concerning the campers and the loaves last week and the corresponding data this week. Above the line, on the left, there were 10 campers arranged in three rows (4 campers in the first and 4 in the second row and 2 campers in the third row) and, on the right, there were 8 loaves in three rows (3 loaves in the first row, 3 in the second and 2 in the third row). Below the line, on the left, there were 15 campers in four rows (4 campers in each of the first three rows and 3 campers in the forth row) and on the right there was a question mark.

As regards the analysis of the data, the following criteria were taken into account for determining the effect of the pictures on MPS: i) whether students used the accompanied picture by acting on it or by observing it in order to solve the problem, ii) whether they confronted internal conflicts by using the picture, and iii) which method/s (visual, analytic or both) they used for the solution of each problem. In regard with the communication process, it was examined whether students, who articulated the problem to their partners, used elements of the picture in their verbal description.

RESULTS

The effect of the decorative picture on MPS was found to be insignificant, since it did not provide any information or feedback for the solution of the problem. All the students overlooked the use of the particular picture in their solution procedures. In particular, six students used analytic methods, whereas two students applied visualization strategies, by drawing their own pictures or diagrams, which seemed to be more helpful than the decorative picture. For example, S8 constructed five rectangles representing the groups, and wrote inside each one of them the numbers 3 and 2, referring to the girls and the boys, respectively. The following statement made by this student indicates the trivial role of the decorative picture: "*The picture of the problem didn't help me at all. My drawing helped me more. By drawing the groups, it was easier for me to find the number of boys and girls*".

On the contrary, the effect of the informational picture was fundamental in problem solving, since it consisted of all the data needed for the solution of the problem and, therefore, was used by all the students. The most common method for the solution of this problem was the trial and error strategy, which was employed by seven students. This strategy involved the active use of the picture as a representational aid, which assisted students to organize their thinking, for example, by noting down next to each item a number indicating its weight. Thus, the use of the picture did not just provide the data of the problem, but it also supported the formation of conflicts between the number combinations proposed by the students, and the correct answers. The only student (S6) who applied a systematic method to find a solution for the problem, by making use of the picture, stated: "*It (the picture) helped me find that one cube and one sphere weigh 11 kg. At the first scale the cube and the sphere weigh 11 kg, so the tetrahedron weighs 12 kg. As for the third scale, since the tetrahedron is 12 kg, 12 plus 12 equal 24, so 4 kg are left, and that's the sphere.*" S2 was the only student who used, almost exclusively, the analytic method to resolve the particular problem. By listening to the description of the problem, she wrote down only the indication of each scale. Then, she reached successfully to a solution by applying mentally the trial and error method, without using any picture. Thus, S2 used the informational picture only for collecting the data needed for the solution and not as a means of representation, or feedback for the solution process, like the other students.

The representational picture was used in an active manner by all the students in problem solving. This is attributed to the spatial ability that was essential for the understanding and the solution of the particular problem. All the students employed visual methods by using the given picture, which in most cases caused internal conflicts. Specifically, the initial intuitive way of thinking before the presentation of the picture, involved the application of the operation of division 10:2=5. By using the picture, students could visualize (actively) the spatial arrangement of the trees at the path, which enabled them to figure out the correct number of trees. In particular, seven students drew five trees at the path and they, subsequently, realized that the distance between the trees was not 2m. As a consequence, they ended by drawing four trees along the path, which was the correct solution. The positive influence of the representational picture was evident in the communication between the students, as well. In particular, during a communication experiment, in the description of the problem to her partner (S2), S1 altered, non-consciously, the data of the problem: "*A man planted 2 trees at each end of a path. Then he planted a tree every 2 m. The length of the path is 10 m. How many trees did the man plant at the path altogether?*" Before the presentation of the picture, based on the incorrect data, S2 conceived the path as a rectangle and doubled the trees of each side. By observing the picture given to her, S2 managed to resolve the problem correctly, without any difficulties. Thus, it can be asserted that the picture facilitated the communication between the students.

The organizational picture had also an important role in problem solving. The use of the picture enabled five students, who had employed additive strategies or inappropriate multiplicative strategies before the presentation of the picture, to use

proportional reasoning by applying correct multiplicative structures. Students who experienced the particular internal conflict made use mostly of visual methods. In particular, they acted on the picture by reorganizing the elements of it, either mentally or practically, so that the 15 campers below the line would be separated into two groups, one of 10 campers and another one of 5 campers, which would be matched with the corresponding groups of loaves (8 and 4). For example, S1 who applied the particular procedure, explained: "*10 campers get 8 loaves, so 10 of the 15 campers would also get 8 loaves. 5 campers are left. I'm thinking that if 10 campers get 8 loaves, then 5 campers get half as much that makes 4 loaves".* Nevertheless, for S5 the picture created an obstacle. Influenced by the arrangement of the elements of the picture, he applied an inappropriate pattern in his solution process. Specifically, he stated that 4 campers have 3 loaves next to them, thus the number of loaves could be found by subtracting 1 from the number of campers (4-1=3). He subsequently applied the pattern to all the rows of the 15 campers and ended with a wrong solution. Three students focused on analytic methods, since they used directly the correct proportion in their solution processes, without using the picture.

Students' views in regard with the role of the pictures in MPS depended on the pictures' contribution in their solution procedures. Specifically, all students stated that the decorative picture could not help in any way; on the contrary, they all recognized that the representational picture had a supportive role in their solution procedures. As for the informational picture, only some students realized that without the picture they could not begin the solution of the problem. This is attributed to the fact that, at first, the picture was presented "orally" to them, so they initially dealt with the problem without using the picture visually. Finally, only the five students who used it in their solution procedures acknowledged the positive effect of the organizational picture on MPS.

In regard with the effect of the different categories of pictures to the communication process, it can be asserted that the need for describing the decorative, representational and organizational picture was trivial. This was evident from the fact that students articulated the problem to their partners, without any reference to the particular types of pictures involved. On the contrary, the description of the informational picture contributed significantly to the communication process, since it provided all the data of the problem.

DISCUSSION AND IMPLICATIONS

Findings of the present study revealed that the representational, informational and organizational picture, but not the decorative one, had a significant effect on MPS. The use of the pictures leaded frequently students to internal conflicts, which in turn enabled them to find correct solutions for the problems. The decorative picture did not have a substantial role on students' responses in MPS or the communication process. This finding gives support to Carney and Levin's (2002) view, that decorative pictures do not enhance the understanding or any application to the text. It

was also found that the arrangement of the elements of the picture may cause difficulties in MPS. This finding is in line with the argument of Colin et al. (2002), who point out that the spatial position of the elements of a picture, may lead individuals to inappropriate inferences.

Most students' strategies involved the use of either visual or analytic methods, focusing on the method that better matched their abilities and the problem or picture involved. The visualization process which was made explicit in two different ways, with students' interaction with the given picture or the generation of a drawing by them, was meaningful and facilitated significantly MPS. Students interacted with the given pictures in two ways, by observing them or by acting on them (Thomas, & Yoon Hong, 2001). Observation was detected mostly in the case of the decorative picture, while active processing was identified mainly in the case of the representational picture. Interaction with the informational and the organizational picture varied; it was active for some students and observational for others. The observed diversity of students' responses concerning the visualization process may be attributed to the interaction between the student and the stimulus, which in the present study refers to the problem and the picture involved. This interactive relationship depends on students' preferences, and their spatial and visualization abilities, i.e. their competence to recall, generate, choose and operate appropriately with the visualization (Bishop, 1989).

As for the communication process, the primary means of communication was the verbal use of language. The organizational and representational pictures had a helpful, supportive and complementary role, and aided in providing precisions, details and feedback for the text and its oral description. The decorative picture had no effect on communication, whereas the informational picture was not only useful, but also essential and substantial in problem solving.

To sum up, from the present study, it is clear that effective problem solving, which makes use of pictures, depends on the relationship between the picture and the problem (function the picture serves in the problem) and the students' previous knowledge and abilities. Therefore, it would seem important that teachers select pictures based on their function in MPS. Another implication for teachers is to take into consideration the students' preferences for the method/s they use in MPS, in order to encourage them to develop the strategies they are not sufficiently competent at. Moreover, findings concerning the function of decorative pictures in MPS underline that the use of decorative pictures in MPS should be handled with great attention in regard with the development of mathematics textbooks, materials, resources and instruction.

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MATHEMATICAL MODELLING WITH YOUNG CHILDREN

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This paper addresses the first year of a three-year, longitudinal study which introduces mathematical modeling to young children and provides professional development for their teachers. Four classes of third-graders (8 years of age) and their teachers participated in the first year of the program, which involved several preliminary modeling experiences followed by two comprehensive modeling problems over a span of 6 months. Regular teacher meetings involving preparatory workshops as well as reflective analyses were conducted. Analyses of children's responses (in group situations) to one of the modeling problems shows the spontaneous ways in which children engaged in sense making, problem posing, hypothesizing, and mathematizing (including representing). It is argued that modeling tasks of the present type are powerful vehicles for developing important mathematical ideas and problem-solving processes in the early school years.

INTRODUCTION

Children today are facing a world that is shaped by increasingly complex, dynamic, and powerful systems of information in a knowledge-based economy (e.g., sophisticated buying, leasing, and loan plans that appear regularly in the media). Being able to interpret and work with complex systems involves important mathematical processes that are under-emphasized in numerous mathematics curricula, such as constructing, explaining, justifying, predicting, conjecturing and representing, together with quantifying, coordinating, and organising data. Dealing with such systems also requires students to be able to work collaboratively on multi-component projects in which planning, monitoring, assessing, and communicating results are essential to success (English, 2002; Lesh & Doerr, 2003). The primary school is the educational environment where all children should begin a meaningful development of these processes and skills (Jones, Langrall, Thornton, & Nisbet, 2002). However, as Jones et al. note, even the major periods of reform and enlightenment in primary mathematics do not seem to have given most children access to the deep ideas and key processes that lead to success beyond school.

In the study addressed here, we introduced 8-year-olds to model-eliciting activities that focus on structural characteristics of phenomena (e.g. patterns, interactions, and relationships among elements) rather than surface features (e.g. biological, physical or artistic attributes). Our longitudinal study addresses both the children's and teachers' growth across three years; however, for this paper we consider the development of the children's mathematization processes as they worked their first modeling activity. One of our initial goals was to introduce the children (and their teachers) to mathematical modeling. To this end, we developed and implemented a number of preliminary modeling activities, which we describe later.

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MATHEMATICAL MODELLING FOR YOUNG LEARNERS

Traditionally, students are not introduced to mathematical modeling until they reach secondary school (Stillman, 1998). However, the rudiments of mathematical modeling can and should begin much earlier than this, where young children already have the foundational competencies on which modelling can be developed (Diezmann, Watters, & English, 2002; Lehrer & Schauble, 2003; NCTM, 2000). Young children's problem-solving experiences have traditionally been limited to problems in which children apply a known procedure or follow a clearly defined pathway. When it comes to solving these problems, the "givens," the goal, and the "legal" solution steps are usually specified unambiguously—that is, they can be interpreted in one and only one way. This means that the interpretation process for the child has been minimalised or eliminated. The difficulty for the child is simply working out how to get from the given state to the goal state. While not denying the importance of these existing problem experiences, they do not address adequately the mathematical knowledge, processes, representational fluency, and social skills that our children need for the 21st century (English, 2002; Steen, 2001). An important component of problem solving in today's world is interpreting the problem situation, dealing with ambiguous or incomplete information, identifying constraints on solutions, and visualising and evaluating possible end-products.

Mathematical modeling takes children beyond basic problem solving where meaning must be made from symbolically described word problems, to authentic situations that need to be interpreted and described in mathematical ways (Lesh, 2001). At the same time, these modeling activities encourage multiple solution approaches and call for multifaceted products. Another important feature of these activities is that key mathematical constructs are embedded within the problem context and are elicited by the children as they work the problem. Children not only have to work out how to reach the goal state but also have to interpret the goal itself as well as all of the given information, some of which might be displayed in representational form (e.g, tables of data). Each of these aspects might be incomplete, ambiguous, or undefined; furthermore, there might be too much or too little data, and visual representations might be difficult to interpret (as in real-world situations). When presented with information of this nature, children might make unwarranted assumptions or might impose inappropriate constraints on the products they are to develop.

Modeling activities for children are also social experiences (Zawojewski, Lesh, & English, 2002) and are specifically designed for small-group work, where children are required to develop explicitly sharable products that involve descriptions, explanations, justifications, and mathematical representations. Numerous questions, conjectures, conflicts, revisions, and resolutions arise as children develop, assess, and prepare to communicate their products. Because the products are to be shared with and used by others, they must hold up under the scrutiny of the team members.

RESEARCH DESIGN AND APPROACH

The study is using multilevel collaboration, which employs the structure of the multitiered teaching experiments of Lesh and Kelly (2000), and incorporates Simon's (2000) case study approach to teacher development. Among the features of multilevel collaboration is a focus on the developing knowledge of all participants, who work as co-investigators operating at different levels of learning (English, 2003). At the first level of learning, children work on sets of modelling activities, which engage them in constructing, refining, and applying mathematical models. At the second and third levels respectively, research students and classroom teachers work collaboratively with the researchers in preparing and implementing the child activities. At the fourth level, the researchers observe, interpret, and document the knowledge development of all participants. Multilevel collaboration is most suitable for this study, as it caters for complex learning environments undergoing change, where the mechanisms of development and the interactions among entities are of primary interest (Salomon, Perkins, & Globerson, 1991).

Participants: Four third-grade classes (8 years old) and their teachers participated in the first year of the study. The classes represented the entire cohort of third graders in a school situated in a middle-class suburb of Brisbane, Australia. The school principal and assistant principal provided strong support for the project's implementation. They were informed of the progress of the study, and attended some of the workshops and debriefing meetings that we conducted with the teachers.

Procedures and activities: We conducted two half-day workshops with the four participating teachers in term 1 to introduce them to the modelling experiences and to plan more thoroughly the year's program. Two more workshops were conducted during the middle and at the end of the year for planning and reflective analysis of the children's and teachers' progress. Several shorter meetings were also conducted throughout the year, including those before and after the teachers had implemented each activity. The contexts of the modelling activities were designed to fit in with the teachers' classroom themes, which included a study of food, animals, and flight.

Preliminary modelling activities were implemented by the teachers towards the end of first term and part of second term. These activities were designed to develop children's skills in: (a) interpreting mathematical and scientific information presented in text and diagrammatic form; (b) reading simple tables of data; (c) collecting, analysing, and representing data; (d) preparing written reports from data analysis; (e) working collaboratively in group situations, and (f) sharing end products with class peers by means of verbal and written reports. For example, one activity involving the study of animals required the students to read written text on "The Lifestyle of our Bilby," which included tables of data displaying the size, tail length, and weight of the two types of Bilbies. Children answered questions about the text and the tables. In another activity focussing on food, the children read about the development of

chocolate from the growth of the cocoa bean to the manufacture of various types of chocolates. Tables of data on the ingredients found in various chocolate types were included. After answering a number of questions on the given information, the children implemented their own survey about chocolate consumption, gathered and analysed the data, and then reported their findings to their peers.

During the remainder of second term and for all of third term, the teachers implemented, on a weekly basis, the main model-eliciting activities. Each lesson was of 40 minutes duration. After an initial whole class introduction, the children worked independently in groups of 3 to 4 to complete the activity. The first model-eliciting activity, which we address here, was titled "Farmer Sprout." It comprised a story about the various types of beans the farmer grew, along with data about various conditions for their growth. After responding to questions about the text, the children were presented with the "Butter Beans" problem. Here, the children had to examine two tables of data displaying the weight of butter beans after 6, 8, and 10 weeks of growth under two conditions (sunlight and shade; see Table 1).

Table 1. Data presented for the Beans Problem (Students were advised that the farmer had grown 4 rows of butter beans under two light conditions)

Sunlight			
Butter Bean Plants	Week 6	Week 8	Week 10
Row 1	9 kg	12 kg	13 kg
Row 2	8 kg	11 kg	14 kg
Row 3	9 kg	14 kg	18 kg
Row 4	10 kg	11 kg	17 kg

Shade			
Butter Bean Plants	Week 6	Week 8	Week 10
Row 1	5 kg	9 kg	15 kg
Row 2	5 kg	8 kg	14 kg
Row 3	6 kg	9 kg	12 kg
Row 4	6 kg	10 kg	13 kg

Using the above data, the children had to (a) determine which of the conditions was better for growing butter beans to produce the greatest crop. In a letter to Farmer Sprout, the children were to outline their recommendation and explain how they arrived at their decision; (b) predict the weight of butter beans produced on week 12 for each type of condition. The children were to explain how they made their prediction so that the farmer could use their method for other similar situations. On completion of the activity, each group reported back to the class.

All whole class interactions were videotaped, while the group work was videotaped (one group per classroom) and audiotaped (2 groups per classroom). We also took field notes of all the class and group activities. The teacher meetings were audiotaped and transcribed, along with the class and group activities.

SOME FINDINGS: CHILDREN'S RESPONSES TO THE BEANS PROBLEM

Prior to addressing some findings, it is worth commenting that each of the teachers established classroom norms for group work and for class reporting. For example, each group of children had a group-appointed manager who was responsible for organising materials and keeping the group on task. The importance of sharing ideas as well as explaining answers was also emphasised. First we consider some general observations of the children's progress on the Butter Beans Problem and then examine the work of one group of three children.

In general, we noted an initial tendency for the children to want to record an answer from the outset, without carefully examining and discussing the problem and its data. The children had to be reminded to think about the given information and share ideas on the problem prior to recording a response. We also observed the children oscillating between analysing the data and discussing at length the conditions required for growing beans. At times, the children became bogged down in these practical discussions, with their progress slowing as a result. This point is revisited shortly.

We noted at least three approaches that the children adopted in analysing the data in Table 1. The first approach was to focus solely on the results for week 10 and systematically compare rows 1 to 4 for each condition (i.e., compare 13 kg with 15 kg, 14 kg with 14 kg and so on). A variation of this approach was to make the comparisons for each of weeks 6 and 8 as well. A second approach was to add up the amounts for week 10 in each condition and compare the results. A third but inappropriate variation of the last approach was to sum all of the weights in each table and compare the results. As one child explained, “Sunlight has 146 to 118 (shade). So plants are in sunlight.” A further approach (again, inappropriate) was to add the amounts in each row for each condition and compare the end results (i.e., 9kg + 12kg + 13kg for sunlight and 5kg + 9kg + 15kg for shade, and so on).

As the children explored the data, they were looking for trends or patterns that would help them make a decision on the more suitable condition. They were puzzled by the anomalies they found, and resorted to their informal knowledge to account for this:

Students collectively: 10 against 6, 11 against 10, and 17 against 13.

Student 1: So this is obviously better than that, but working out why is the problem.

Student 2: Yes, because the more sunlight the better the beans are. For some reason...

Student 1: In some cases, it’s less; but in most cases, it’s more the same.

Student 3: It would depend on what type of dirt it has been planted in.

Student 2: I’ve got an idea. Perhaps there were more beans in the sunlight.

Student 3: We’re forgetting one thing. Rain. How much rain!

The above group of students spent quite some time trying to suggest reasons for the trends in data and became waylaid by practical issues. Nevertheless, in doing so, the children engaged in considerable hypothetical reasoning and problem posing, for

example: “I’ve got an idea. If we didn’t have any rain, the sunlight wouldn’t …it wouldn’t add up to 17 (kg). And, if we didn’t have any sunlight, it wouldn’t be up to 17 either. But if we had sunlight and rain….”

For the remainder of this lesson, the above group of students cycled through finding practical reasons for *why* they thought sunlight was better, re-examining the sets of data, and attempting to record their findings. The children did not make substantial progress during this lesson largely because of their focus on finding reasons for the trends in the data. As one child explained to the researchers, “But our problem is, we thought it would be because of the rain. It can’t get in as well with the shade cloth on. But then we found these results. And we’ve got a problem. We can’t work out why this has popped up. So we’re stuck here.”

Considerable progress was made in the second lesson, however, after one of the children (Amy) directed the others to focus solely on the data. The child directed their attention to row 3, week 10, where the difference was the greatest (“Here’s the best and here’s the worst”). Amy attempted to show the other group members this difference by drawing a representation of the two amounts. Her representation took the form of a simple bar graph (“picture graph”, as she described it) with the first bar coloured yellow to represent the 18kg (sunlight) and the second bar coloured black to represent the 12kg (shade). In directing her peers’ attention to her diagram, Amy explained, “O.K., so that would be about the sunlight there…..what I am trying to say is the shade is about half as good as sunlight.” Her peers, however, were not paying attention so she decided to pose this question to bring them back on task: “This here is sunlight and this here is shade. Which one’s better?” Still not happy with her peers’ lack of enthusiasm, Amy posed a more advanced question for her peers:

Amy: Oscar, if this long piece was shade, and the short piece was sunlight, and they represented the weight of the beans, which one would be better?

Oscar: This.

Amy: No, shade would be because it’s bigger. A bigger mass of kilograms.

The difficulty for many of the children was completing the letter for Farmer Sprout. As Amy explained to the teacher, “You see, I’ve drawn a picture graph and we’ve worked out the answer, but we can’t put it into words... I know! We can draw this (her representation) on our letter and explain what it means in words. And that’ll get us out of it.” The group finally produced the following letter, choosing to focus solely on the largest difference between the conditions:

> Dear Farmer Sprout, We have decided sunlight is the best place to grow Butter Beans. Because if this was the best (*an arrow pointing to the representation*) and this the worst (*another arrow pointing to the representation*), black = shade and yellow = sun. 18kg and 12kg. It is obvious that sunlight is better because 18 is higher than 12 by six. We came to this decision because sunlight maenley (sic) projuced (sic) more kg or the amount of kgs. Yours sincerely, Mars Bars (*name of the group*).

In reporting their findings to the class, the above group commented that “Shade produced about half as much as sunlight altogether.” When asked where they obtained their information for this conclusion, Amy explained, “Well, we basically

added all of this up (week 10 data for each condition) and we found that shade produced about half as much as sunlight altogether."

In responding to the second component of the Butter Beans Problem, the children generally relied on patterns in the data to predict the mass of the beans after 12 weeks. For example, another group in Amy's class reported their predictions for the sunlight condition as follows: "Our findings show that in row 1, week 12, you will get 15 to 17 kilograms, and in week 12, row 2, you'll get 17 kilograms, and in row 3, week 12, you will get 19 to 21 kilograms, and in week 12, row 4, you shall get 18 to 20 kilograms. That's what we think for sunlight." When asked how they got these findings, the children explained, "The data, because we went to week 10 and we counted 2 on….because they've sort of gone up like, in twos and it was another two." It is interesting to note that in one class discussion on the likely mass of the beans at week 12, some children again extrapolated beyond the data and claimed that at week 12 the beans would be too old and would probably die.

CONCLUDING POINTS

The model-eliciting activities used in our study encourage young children to express ideas related to structural characteristics of meaningful phenomenon. We saw varying levels of sophistication in the mathematization processes of the children. There appeared to be several elements that either facilitate or constrain the growth of these processes. First is the ability to interpret and understand data presented in various representational formats. Although the majority of children had few difficulties here, we observed a few children who misinterpreted what was being measured (i.e., they thought that the number of kilograms referred to the amount of sunlight rather than the mass of beans). A second element was an intuitive knowledge and understanding of the mathematical concepts inherent in the task, such as the notions of change and rate of change. Intuitive notions of aggregating and averaging were expressed by some of the student groups. A third element was the issue of existing personal knowledge versus task knowledge. Being able to distinguish between the two knowledge forms, and knowing how they can both help and hinder solution facilitate task success. We see the development of such metacognitive and critical reasoning skills as important components of these modelling activities. Fourth, the modelling tasks provided rich opportunities for children to express ideas in multiple representations. We saw how children used text, diagrams, and verbal explanations to engage in effective learning in a socially mediated environment.

There appear to be several implications for teachers. First children who engage in social learning contexts verbalise their thinking and hence make explicit their knowledge of content and processes. Opportunities thus exist for teachers to extend children's knowledge in areas of need. At least one of the participating teachers often capitalised on discussions to make links to historical information that the children may have previously learned. Some links were made to scientific or social studies but few explicit connections were made to mathematics. The modelling activities were embedded in a broader theme being explored by the class but explicit links were

often not taken up by the teachers who appeared to assume that the activity needed to be isolated for the benefit of the researchers. Opportunities for children to actually implement the activity described in the modelling task can help young children who might lack some of the related conceptual knowledge. In the present case, undertaking the task of comparing growth conditions of plants (either in real life or via computer simulation) can provide concrete representations that facilitate model development.

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EXTENDING LINEAR MODELS TO NON-LINEAR CONTEXTS: AN IN-DEPTH STUDY ABOUT TWO UNIVERSITY STUDENTS' MATHEMATICAL PRODUCTIONS[1]

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This research report presents a study of the work of agronomy majors in which an extension of linear models to non-linear contexts can be observed. By linear models we mean the model y=a.x+b, some particular representations of direct proportionality and the diagram for the rule of three. Its presence and persistence in different types of problems and teaching contexts have drawn us to search for alternative explanations; we employ a qualitative methodology using individual interviews and students' mathematical written tasks. The data allowed us to make an in-depth descriptive analysis of students' strategies when solving non-linear problems and their reasons to decide the model to be applied.

THEORETICAL AND EMPIRICAL BACKGROUND

This study is related to the mathematical understanding of agronomy majors from the University of Córdoba (Argentina). It extends a previous research (Esteley, Villarreal and Alagia, 2001) in which we focus on the documentation, description and analysis of a phenomenon that occurs among these students, which we denominate *extension of lineal models to non-linear contexts* or *overgeneralization of linear models*. By linear models we mean the model $y=a.x+b$, some particular representations of direct proportionality and the diagram for the rule of three. Such a phenomenon occurs when the resolution of certain mathematical questions relating two variables, is solved applying linear models, even though the situation, from the teacher's point of view, is obviously a non-linear one. The presence of this phenomenon doesn't necessarily imply that the students are conscious that they are applying linear models in non-linear contexts.

This phenomenon has been studied with students of the elementary school, with focus on the particular representation of direct proportionality and it is known as *linear misconception*, *illusion of proportionality* or *linearity* and also *proportionality trap* (Behr, Hare, Post & Lesh, 1992). The tendency of overgeneralising the use of linear models beyond its range of validity is also present in secondary school pupils. The extensive studies of De Bock, Von Doorem, Janssens & Verschaffel (2002), De Bock, Von Doorem, Verschaffel & Janssens (2001) and De Bock, Verschaffel & Janssens (1998), carried out with 12-16-year old students, reveal a strong tendency to apply linear models to solve proportional and non-proportional word problems about the relationship between lengths and perimeters/areas/volumes of similar figures or

solids. These authors carried out research based on written tests and some in-depth interviews. The explanation for the illusion of linearity has been investigated mainly from the perspective of the error as a learners' deficiency, and an intuitive approach towards mathematical problems, inadaptative beliefs and attitudes or poor uses of heuristics are indicated as factors of the student's unwarranted proportional reasoning (Van Doren, De Bock, De Bolle, Janssens & Verschaffel, 2003). The literature is extensive reporting on primary and secondary school students' illusion of linearity and there exists agreement in describing the phenomenon as persistent and resistant to change. Nevertheless studies on this phenomenon among university students are not frequent, even though its presence and persistence have been frequently observed at that level within diverse types of problems and contexts. That situation led us to carry out an exploratory study (Esteley et al, 2001) to document, describe and analyze the presence of the phenomenon of overgeneralization of linear models among Argentinean 18-20-year students, which studied Agronomy in the University of Córdoba. We analyzed, through the students' written productions, the types of problems that were solved by extension of a linear model, the strategies followed by the students and the difficulties of interpretation that could be associated with the statements of some problems

At this point, we decided to yield explanations beyond the notion of the error as students' deficiency or fault. Therefore, in our studies, the errors were assumed as symptoms of the conceptions underlying the students' mathematical activities, in the sense of Ginsburg (1977) or Brousseau (in Balacheff, 1984). After our exploratory study, briefly described above, we decided to deepen our investigation, performing interviews with those students that, in our previous studies, had applied linear models to solve non-linear problems. In this sense, the studies of Confrey (1991, 1994) and Confrey & Smith (1994) addressing questions about the epistemological value of the students' mathematical constructions are paramount for us. Confrey (1991) argues that to understand the students' actions implies to be introduced in their perspective and not to presuppose that it coincides with that of the teacher/researcher. The students' answers that stray from the expectation of the teacher/researcher, can be legitimate as alternative or valid and effective in other contexts. To encourage the student to show their points of view implies for the teacher/researcher an opportunity to glimpse at the students' perspectives and to question her/his own, examining them through the ideas of the students.

RESEARCH METHODOLOGY

The research methodology was qualitative (Lincoln & Guba, 1985) since we aimed to get in-depth understanding of the students' thinking processes when they extend linear models to non-linear contexts. Individual semi-structured interviews were performed with 18-20-year old agronomy majors from the University of Córdoba (UNC) that were attending a calculus course. These students had shown the appearance of the phenomenon of interest in our previous studies. We carried out a

single an-hour-long interview with each student. The interviewer was not the students' teacher. All interviews were audio-taped and paper, pencil, and a scientific calculator were at the interviewees' disposal.

The interviews were structured around the activities and aims we describe next. Activity 1) Ask the student to explain the way she/he had solved the Problem A (see Figure 1) in our previous study (Villarreal et al, in press) with the aim of elicit the student's strategies and reasons to apply a linear model in a non-linear context. Activity 2) Propose to solve Problem B (see Fig. 1) with the aim of elicit the student's strategies while solving a new non-linear word problem. Activity 3) Ask to calculate the height of the plant in Problem B at any time with the aim of challenge the student to produce a general model and verify the consistency of the student's strategy. In all the activities we asked the students to think aloud (Ginsburg, Kossan, Schwartz & Swanson, 1982) while solving the problems. The selected problems are typical in the introductory mathematics course for the agronomy majors from UNC.

We did an inductive/constructive analysis (Lincoln & Guba, 1985), since we didn't raise a priori hypothesis, but rather, we generate conjectures from the gathered data. The analysis of the students' strategies and solutions to the problems was not carried out in terms of "right or wrong". Although conceptions not accepted as correct by the mathematicians can be indicated, the emphasis was on students' thinking processes, without making comparisons, but trying to listen closely (Confrey, 1994).

Problem A) *Say if the following statement is true or false and justify your answer. An insect, that weighs 30 gr. when being born, increases its weight at 20% monthly. Then, its weight after two months is 43,2 gr.*

Problem B) *If a plant measures, at the beginning of an experiment, 30 cm and every month its height increases 50% of the height of the previous month, how much will it measure after 3 months?*

Figure 1: Problems A and B

RESULTS AND ANALYSIS

In this report we decided to present the results related to the interviews performed with Santiago and Clelia. In the selected excerpts, we underlined some of the students' assertions to indicate the words we believe support our analysis; we include, between brackets, explanations to better understand the students' expressions or words that give continuity to the text; [...] indicates long pauses, meanwhile short pauses are represented with simple dots.

Santiago

Santiago had solved Problem A as follows: he wrote $y(t) = 30 + 6.t$, he calculated $y(2)$ and finally he answered that the statement was false because "*42 gr ≠ 43.2gr*"

After looking at his written solution, during activity 1, Santiago said: *"I wanted to do it using a function ... and not with a rule of three* [...] *Then, I realized that it wasn't a linear function because that* [he refers to the insect] *hasn't an unlimited growth, it was an exponential or something like that"*. When Santiago rejected his linear solution and gave his justification, it became apparent to us that he was using biological reasons instead of mathematical ones to select a new model. Although Santiago realized that the insect growth was not linear but exponential, he stated: *"When I did realized that it was an exponential, I also realized that I wasn't able how to do it, because, I don't know* [...] *it seems to me that I don't have the tools yet"*.

Santiago considers the rule of three as a mechanical procedure with a low mathematical status, at least, for university students and that is why he decided to use the linear function $y(x) = a.x + b$. We can also point out that the student use biological reasons to reject his initial linear solution and try a different approach, although he considered conditions not given in the problem statement. We could also recognized these aspects when Santiago solve Problem B), showing a strong consistency.

During activity 2) and, after reading Problem B), the following dialogue occurred:

Santiago: the height of the plant is 30 cm at the beginning of the experiment, so, that is the "base", and each month it grows up 50% of the height it had the previous month …

Interviewer: yes

Santiago: well, I do it with the rul… well in a sort of mechanical way, the first month it would be 30... plus the 50% of 30. [he used the calculator and wrote the first line in Fig. 2] In the second month I start at 45 plus 50% of 45, that would be... [he wrote the second line in Fig. 2] and in the third month I start at 67.5 and I do the same [he wrote the third line in Fig. 2]

30 cm	$\rightarrow 30 + 15 = 45$	*1 mes*	[1st month]
45 cm	$\rightarrow 45 + 22.5 = 67.5$	*2 mes*	[2nd month]
67.5 cm	$\rightarrow 67.5 + 33.75 = 101.25$	*3 mes*	[3rd month]

Figure 2: Santiago's written solution of Problem B

When the interviewer asked how to calculate the height of the plant after twelve months or at any time *t*, the next dialogue took place:

Santiago: I should do it with a function, to make it easier [...] it has to be an exponential and... because the growth has to be like this [he makes a gesture with his hand indicating an S- like line], it cannot grow up indefinitely… besides, because the variable is changing.

Interviewer: what do you mean when you say "the variable is changing"?

Santiago: because when I get to the first month, I see that it changes, I stop working with 30 and I start working with 45 and then, I change from 45 to 67… It is like that it is always moving beyond. I should see if there is any ..., no, I don't know if it is possible to have a relationship of growth, no, no, no ... if the 45 has the same increment proportion with this… but not… that from 30 to 45 it jumps the same, no, neither here, no, I don't know.

Interviewer: what do you mean with: “it jumps”?

Santiago explained that he was trying to find a relationship between the height of the plant for each month. He talked about a "parameter" that would show the growth. In order to get it he drew a line as the one sketched in Fig. 3. The student started searching for that "parameter" calculating the following differences: *45 - 30; 67.5 - 45* and *101.25 - 67.5*. Santiago wrote those differences in the second row of numbers in the Fig. 3. He indicated that the increment from 30 to 45 was not the same as that from 45 to 67.5 and from 67.5 to 101.25. After that, he continued searching for the "parameter" but, this time, doing the differences between the values in the second row. He calculates *22.5 - 15* and wrote *7.5* on third row (see Fig.3). While working with the calculator, he realized that adding *7.5* to *22.5* he wouldn't obtain *33.75*. After a while, he gave up this strategy and said he should have to look for information in a textbook where a solution of a similar problem could appear.

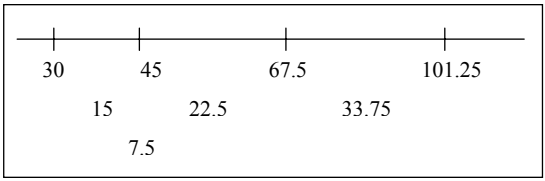

Figure 3: Sketch of Santiago's line

Therefore he tried with other models. Firstly, he drew an upward pointing parabola and immediately rejected it because of its *"unlimited increasing"* and *"time couldn't be negative"*. Then, he proposed $y = a^x + b$ and when the interviewer asked him for the value of b, he said it would be *"the initial 30 cm"* but, he finally said: *"That one* [he referred to $y = a^x + b$] *doesn't help me, either, since it also has an unlimited growth, it must be something tending to a number... something that bends down* [he drew an increasing graph with an asymptote].

The graphical representations of a limited growth that Santiago considered are consistent with the biological conditions he added to the problems. We should point out that his option for searching an additive constant (the "parameter") to model the variation of growth finally became an obstacle for him.

Clelia

Clelia solved problem A as follows: she wrote the initial weight of the insect as p_i=*30gr*, she calculated the 20% of p_i using a rule of three (see Fig. 4) and continued working as it is shown in Fig. 4.

30 gr.----------100% p_i *6gr. = x------- 20% p_i* *1 mes pesa 30 + 20 % de 30 = 3* *2 meses pesa = (30 + 20 /100) 2 + 30 gr* *12 gr. + 30 = 42 gr*	[In 1 month weights...] [In 2 months weights...]

Figure 4: Clelia's written solution of Problem A

Finally, she concluded that the statement was false since after two months the insect would weigh 42 gr. From Clelia's written solution we can infer that, for her, *(30 + 20/100)* is equivalent to *6*, and that the implicit model for the insect weight, behind her calculations, is *w(t)= 6.t + 30*. After observing her written solution, Clelia said: *"I thought on it and then, talking with my classmates, we realized that we had to calculate the 20% of the weight, month by month, while the weight was increasing, not to the initial weight... we added in that way"*. Clelia referred to the fact that she was assuming that the insect always grows 6 gr. every month.

During activity 2), after Clelia had read Problem B), the following dialogue occurred:

Clelia: After a month it will measures 30 [...] plus 50% .... of 30... This is for one month, and for three months [...] all this times three [she laughed]

Interviewer: please, continue

Clelia: I should calculate one by one to make it easier... 50% of 30 [she made a pause while whispering something inaudible and stopped]

Interviewer: you have just said "all this times three", what do you mean by "all this"?

Clelia: no, it just seemed to me

Interviewer: it doesn't matter, when you said "all this", what were you referring to?

Clelia: To the 50% of 30 plus 30

At this point, Clelia gave up her approach, probably because it was the same that she had used to solve Problem A, and she knew that it wasn't right. After a while, she started using a calculator and finding a value for each month as it is shown in Fig. 5.

1er. Mes $30 + \frac{50}{100} 30 = 45$	[1st month]
2do. Mes $30 + \frac{50}{100} 45 = 52{,}5$	[2nd month]
3er. Mes $30 + \frac{50}{100} 52{,}5 = 56{,}25$	[3rd month]

Figure 5: Clelia's written solution of Problem B

When the interviewer asked Clelia to calculate the height of the plant after twelve months she answered: *"I don't know. We know that plants don't grow unlimited, at*

some moment they stop growing... Well, I would calculate the 50% of the height from the previous month and add [such percentage] *to it* [referring to the previous height of the plant]. After talking about this particular case, Clelia wrote the following formulae as a model for the growth of the plant:

$$h = 30 + (\frac{50}{100}\, x) \qquad x = \textit{altura del mes anterior}$$ [height of previous month]

In this way, Clelia wrote down an expression in which the height of the plant for each month was calculated from its height in the previous month. Finally she couldn't get a height-time expression.

The interviewer asked to Clelia what she was thinking about when she spontaneously took the decision of always adding 30 cm to the percentages she had been calculating while solving Problem B (see Fig. 5), and she answered:

Clelia: The existence of a constant [with emphasis in her voice]. It begins here, and <u>to this one... I have to add</u>

Interviewer: Is that what you were thinking on?

Clelia: Yes, I always notice that [with strong emphasis in her voice]

We would like to point out the strong and consistent presence of an initial value (30 in both problems) to which the student always adds the variation in and weight and height.

CONCLUSIONS AND DISCUSSION

The interviews provided us with relevant information about the strategies and thinking processes followed by the interviewees while working with non-linear problems. In that sense we point out that Santiago and Clelia share some important aspects. Both students chose a linear model to solve problems A, but while Santiago explicitly said that he had applied a linear function, Clelia didn't say anything about the kind of model she had applied. When the interviewer challenged them to find a general model both students regarded the growth (for the insect or the plant) as an "additive model" in the form of 30 + (something variable in time). We called it additive model since the initial value (either the weight or the height) is always added to the next variation. For example, in the case of Santiago he considered $y(t) = 30 + 6.t$ in Problem A or $y(t) = a^t + 30$ in Problem B. In the case of Clelia she proposed a recursive relation that enables her to calculate the height of the plant (of Problem B) at time t adding to 30 cm its height at time $t-1$. Both students introduced biological constraints (unlimited increasing), external to the statements of the problems and proposed graphical models that were consistent with those biological constrains. The additive model and the external conditions finally became obstacles for obtaining the general model that accounts for the situation posed in activity 3. We must recognize that the students were able to establish connections with reality, which is considered a positive habit for agronomy students. Finally we want to point out that although the students had the mathematical tools necessary to construct a general algebraic model,

they couldn't get it. This last observation with the results reported suggest that the overgeneralization problem is not a trivial one, goes beyond school levels and needs a thorough investigation.

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UNIVERSITY STUDENTS' CONCEPTIONS OF FUNCTION

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"Function", as it is understood today, formulates one of the most important concepts of mathematics. Nevertheless, many students do not sufficiently understand the abstract but comprehensive meaning of function and problems concerning its didactical metaphor are often confronted. The present study examines the interpretation of the concept of function among second year students of the Department of Education, at the University of Cyprus, and outlines their misunderstandings and possible obstacles in fully grasping its meaning. Results have shown that students' perception of function appears in isolated components of mathematical ideas associated with the concept of function.

INTRODUCTION

A historical perspective of the way the concept of "function" came to exist in contemporary mathematics would reveal centuries of discussions among mathematicians. On the other end, the didactical metaphor of this concept seems difficult, since it involves three different aspects: the epistemological dimension as expressed in the historical texts; the mathematics teachers' views and beliefs about function; and the didactical dimension which concerns students' knowledge and the restrictions implied by the educational system. On this basis, it seems natural for students of secondary education, in any country, to have difficulties in conceptualizing the notion of function.

The present work examines the interpretation of the concept of function by second year students of the Department of Education, at the University of Cyprus. Since the participants come from different secondary school directions, the present investigation is likely to reveal various types of misunderstandings. Predominantly, these students are prospective primary school teachers, who will in a way transfer their mathematical thinking to their future students.

EPISTEMOLOGICAL DIMENSION AND THE DIDACTICAL METAPHOR OF FUNCTION

The concept of function is central in mathematics and its applications. It emerges from the general inclination of humans to connect two quantities, which is as ancient as Mathematics. Nevertheless, what directed to the idea of managing unique relations, which is accepted in the formal definition of function, was the need for calculations, within the framework of Analysis, especially during modernity. Based on the definitions of Euler, Bernoulli, and Cauchy, Dirichlet in 1837, concluded in the expression *"Variable y is said to be a function of variable x defined in the*

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interval a<x<b, if to every value of variable x in this interval corresponds only one value of variable y, independently from the form of the correspondence". In this definition the concept of variable includes a timeless intelligible election of value within the space of real numbers. The set-theoretic definition of Dedeking was the next stage (Davis J.P., & Hersh R., 1981).

Consequently, to sum up, function, as a typical mathematical concept, is a mental construction that was integrated rather recently in mathematics. It is a matter of synopsis and congregation of different experiences and conceptual tools that mathematicians and scientists initially used to solve problems and assemble theories.

Due to this historical concentration, the notion of function is so abstract that presents many difficulties in its didactical metaphor. Different epistemological approaches that led to the meaning of function through its long historical evolution are disrupting into the teaching guides and textbooks of mathematics in a confusing way. The complexity of this didactical metaphor has been a main concern of mathematics educators and an active question in the research of mathematics education (Dubinsky & Harel, 1992; Sierpinska, 1992; Gomez & Carulla, 2001; Hansson & Grevholm, 2003). Moreover, the understanding of functions does not appear to be easy, because of the diversity of representations associated with this concept, and the difficulties presented in the processes of articulating the appropriate systems of representation involved in problem solving (Yamada, 2000). Therefore, a substantial number of research studies have examined the role of different representations on the understanding and interpretation of functions (Thomas, 2003; Zazkis, Liljedahl, & Gadowsky, 2003).

Researchers usually investigate the epistemological obstacles, on the basis of the historical study of the concept of function, and propose teaching methods, which aim at overcoming these obstacles. In practice, different approaches that are applied in mathematics instruction concerning the concept of function result in exposing to the students the pieces of a puzzle consisting of a vague set of extracted information, that possibly merge at university level in mathematics. Sierpinska (1992) gives a viable example of such an approach supporting that formulae, graphs, diagrams, word descriptions of relationships and verbal expressions, compose an uncertain schema of thoughts.

We believe that further research regarding the understanding and use of functions by university students is needed, so that their difficulties and misconceptions are identified. This could lead to planning and applying appropriate and efficient instruction at university level, for improving students' comprehension about functions. The present study aims to provide answers to the following research questions: a) How do students conceive and use the concept of function? b) How do students recognize functions in multiple representations? It should be noted that the main concern of the present study is beyond the measurement of the success rate to the proposed tasks, and focuses on the connections of students' conceptions about functions, as indicated by their responses to the tasks.

METHOD

The sample of the study consisted of 164 students who attended the course "Contemporary Mathematics" during the first semester of the academic year 2003-2004, at the University of Cyprus. The course is compulsory for the students of the education department, and can be elected by the students of mathematics department students. The questionnaire was completed by 154 second year students of the education department and 10 four year students of the department of mathematics.

Students were asked to complete a written questionnaire that included tasks of recognition of functions among other forms, given in various types of representation (verbal expressions, graphs and mapping diagrams or algebraic expressions). A variety of functions were used: linear, quadratic, discontinuous, piecewise and constant functions. Furthermore, students were asked to provide a definition of what function is and two verbal examples of functions application in real life situations. Below we give a brief description of the questions:

Question 1: Recognition of functions between four given verbal expressions (Q1a, Q1b, Q1c, Q1d).
Question 2: Construction of the characteristic function of a set (Q2).
Question 3: Construction of the algebraic expression of a function given in verbal expression (Q3).
Question 4: Recognition of functions between six given graphs (Q4a, Q4b, Q4c, Q4d, Q4e, Q4d).
Question 5: Construction of a graph from an algebraic expression of one of the functions of the previous question (Q5).
Question 6: Recognition of functions between five given graphs (Q6a, Q6b, Q6c, Q6d, Q6e).
Question 7: Construction of a graph of a function with domain distinct points (Q7).
Question 8: Recognition of functions between four given diagram mappings (Q8a, Q8b, Q8c, Q8d).
Question 9: Definition of function (Q9).
Question 10: Examples of functions from their application in real life situations (Q10).

Correct and wrong answers were accounted for all the questions. Answers to questions 9 and 10 were given additional codes as it is further described.

The definitions given by the students were additionally coded as follows:
D1: *An approximately correct definition*. In this group the following type of answers were included: (i) accurate definition, (ii) correct reference to the relation between variables but without the definition of the domain and range, (iii) definition of a special kind of function (e.g. real function, function one-to-one or on to, continuous function).
D2: *Reference to an ambiguous relation*. Answers that made reference to a relation between variables or elements of sets, or a verbal or symbolic example were included in this group.

D3: *Other answers*. This type of answers made reference to sets but without relation, or reference to relation without sets or elements of sets.
D4: *No answer*

As for the examples, they were coded as follows:
X1a: Example of a function with the *use of discrete elements of sets*.
X1b: Example of a *continuous function* from physics
X2: Example of a *one-to-one function*.
X3: Example presenting an *ambiguous relation* between elements of sets.
X4: Example of an *equation* (verbal or symbolic).
X5: Example presenting an *uncertain transformation* of the real world.
X6: *No example*.

For the analysis and processing of the data, Gras' s implicative statistical analysis was conducted by using the computer software CHIC (Bodin, Coutourier, & Gras, 2000). A similarity diagram, which allows for the arrangement of tasks into groups according to their homogeneity, was produced. The notion of 'supplementary variables' was also employed in the particular analysis. Supplementary variables enable us to explain the reason for which particular groups of variables have been created and indicate which objects are "responsible" for their formation. In our study, secondary school direction and field of study (i.e. education or mathematics) were set as 'supplementary variables'. Consequently, we were able to know which school direction or study field contributed the most to the formation of each group.

RESULTS

The results are presented into three sections. In the first section we present some indicative answers given in the last two questions and in the second section we present the percentages of success. In the third section we present the results of the implicative analysis using software CHIC.

(i) Some indicative answers

We restrict the qualitative analysis to the answers given in the last two questions, since they are of most interest.

In the question requiring the definition of function the answers that gave an approximately correct definition were grouped together. "*Function is a relation between two variables so that one value of x (or the independent variable) corresponds to one value of y (or the dependent variable)*" were accounted in this group. Answers like "*Function is an equation with two depended variables*", "*Function is a relation in which an element x is linked with another element y*" or even "*Function is a mathematical relation connecting two quantities*" were coded as D2. As D3 we have coded answers, which made reference to sets, but did not mention relation, or involved relation and not sets or elements of sets, that is answers like "*Function is relation*" or "*Function is a mathematical concept that is influenced by two variables*" or "*Function is the identification of parts of a set*".

The correct examples of a function were of two kinds (X1a and X1b). Examples of the first kind were "*Each person corresponds to the size of his shoes*", "*Each student corresponds to his/her mark at the test*" made use of sets with discrete elements. The second type of examples presented a continuous function mainly from physics such as "*The height of trees is a function of time*", "*Atmospheric pressure is a function of altitude*". The examples presenting a function one-to-one were coded separately as X2. Such answers were "*Every citizen has his own identity number*", "*Every graduate has his own different degree*", "*Every country corresponds to its own unique name*". As X3 we coded the examples presenting a relation between elements or variables but without clarification of the uniqueness in function. Such answers were "*There is a relation between students and their books*", "*The prices of vegetables depends on the production*", "*We correspond the marks of girls in a classroom to those of boys*". Examples presenting an equation instead of a function were coded as X4. "*There are 2x boys and 3y girls in a classroom and all the children are 60. If the boys are 15 we can calculate the number of girls*", "*Kostas has x number of toffees and John has double that number. How many toffees do the two friends have?*". The last category X5 included answers, which were ambiguous and in addition they did not define any variables or sets, and referred to general transformations of real world. Such answers were "*Health depends on smoking*", "*Success in a test depends on the hours of studying*", "*In the relation of children and parents, the children are the depended variable and parents the independent variable*".

(ii) Percentages

For the purposes of the present study we will only refer to the results, which show the strongest trends among the students. Question 1, requiring the recognition of verbal examples of function, was answered successfully by around 50% of the students, and this percentage was almost uniform for all the four parts of the same question. On the contrary, in Question 4 concerning the recognition of function given in an algebraic expression, the percentages varied between the different parts of the same question. The linear function $2x+y=0$ was recognised by 73% while 65% of the students answered that the equation of a circle $x^2+y^2=25$ presents a function. In Question 6, which concerned the recognition of a function when given in a Cartesian graph, the most difficult part was the line $y=4/3$ which was recognised as a function only by the 27% of the students, since it was treated in identical way with the line $x=-3/2$. In the same question, the discontinuous linear function of Q6e was recognised only by 31% of the students. It can be asserted that the majority of students appear to identify the stereotypical forms familiar to them from high school as functions.

(iii) Gras' s Implicative Analysis

From the similarity diagram shown in Figure 1, it ensues that there is a connection between four small groups Gr1, Gr2, Gr3, Gr4 that comprise the bigger cluster A. From these subgroups, the "strongest" is Gr2 formed around variables D1 and X2 that present the premier similarity (0,99999). That means that students who give an approximately correct definition (D1) in Question 9, give an example of a function one-to-one (X2) in Question 10. Around this strong subgroup the answers to questions Q6d and Q6e are linked. These are the questions asking the recognition of

some non-conventional cases of functions presented graphs (Q6d was a graph not representing a function and Q6e presented the graph of a discontinuous linear function). Finally this subgroup is completed with the answers in Question 2 (Q2), which concerns the translation from a verbal representation of a piecewise function to the algebraic form

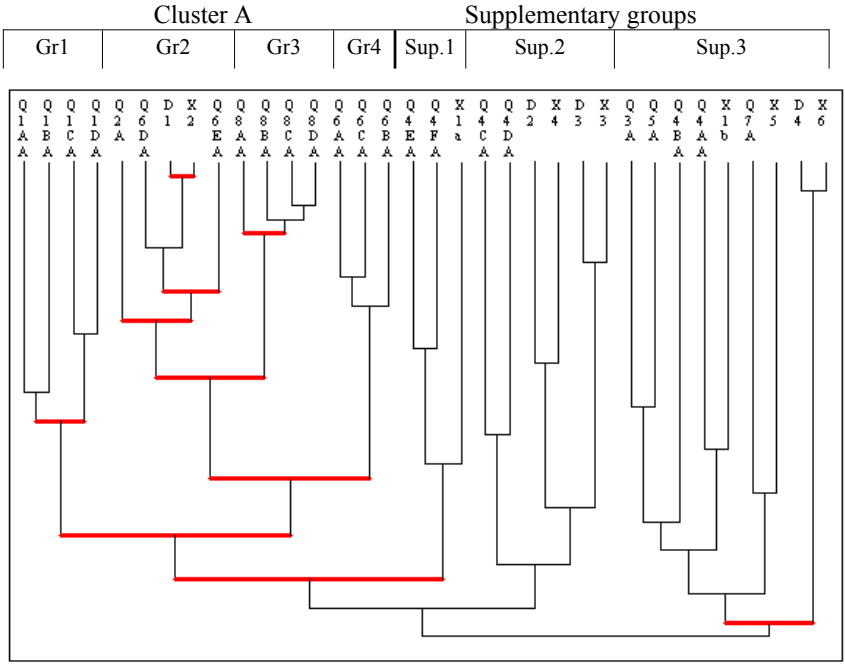

Figure 1: Similarity diagram of the observed variables

Note: Similarities presented with bold lines are important at significant level 99%.

Around the strong group (Gr2) three other subgroups are organised (Gr1, Gr3 and Gr4), which concern the answers to the four parts of Question 8 (Q8a, Q8b, Q8c, Q8d) that is the recognition of functions presented in the form of mapping diagrams. The high similarity of this group (0,997) indicates that mapping diagrams are confronted in the same isolated way. The two groups Gr2 and Gr3 compose a new strong subgroup with degree of similarity 0,899. The subgroup Gr4 is further connected with the strong connection of subgroups Gr2 and Gr3, which include the answers to the other parts of Question 6 (recognition of function given in algebraic form). Conclusively the connection of subgroups Gr2-Gr3-Gr4 creates a group of answers, which show a conceptual approach to function. In other words the behaviour of the students to the definition and to the provision of an example of function has a predictive character in terms of their behaviour to functions when they are represented as graphs and diagrams.

Group Gr2-Gr3-Gr4 connects with Gr1 that includes the answers to Question 1, that is the recognition of functions when they are presented in verbal form. Finally this whole group (Gr1-Gr2-Gr3-Gr4) connects with the most "extraordinary" examples of Question 4 (Q4e and Q4f) that refer to recognition of functions in algebraic form. These are connected with the group that gave a correct example with the use of discrete elements of sets. This is the first "supplement" of group A.

The second supplement of strong connections is embodied by definition D2 and D3 and examples X3 and X4 that illustrate a vagueness or limited idea of the definition and the examples of function. These variables connect with answers to questions Q4c and Q4d, which are treated in that way that shows the wrong belief that in an algebraic form of a function symbols x and y must always appear. The third supplement is the group with the most doubtful idea about the notion of function since it includes D4 and X6 (i.e. those students that did not attempt any definition or example of function). Furthermore, this group is justified from high school direction, i.e. the students who have followed direction of classical studies. The third supplement behaves as an autonomous subgroup and consists of the answers that show absence of definition or example with a group of different questions that all have directly or indirectly a linear-algebraic character (Q4a, Q4b, Q3a, Q5a, Q7a). Also variable X1b (examples of function with the use of discrete elements of sets) is also connected with this group. The students that give answers that belong to the last group appear to have the misconception that function is just a linear relation.

Conclusively the strongest similarities in the diagram are (a) among responses providing correct definition and examples of functions and are mainly attributed to the students of mathematics department and (b) among responses giving no or very ambiguous answers and are attributed to the students of the education department, who come from the classical high school direction.

CONCLUSIONS

The study has revealed three strong trends in the ideas of students for function. The first is the identification of "function" by a large percentage of students with the more specific concept of "function one-to-one". The idea of uniqueness is particularly condensed and leads to identification of function as one-to-one function. Although this idea works for a wide range of situations and problems involving functions, it becomes a strong obstacle for the understanding of function as a wider concept. The second trend is the idea that "function" is an analytic relation between two variables (as it worked historically, initially with Bernoulli's definition, and more clearly with Euler's) and it is apparent in the way students define function and the examples they give. The third trend is that "function" is connected with a kind of diagram, either a Cartesian graph or a mapping diagram. On the contrary, when dealing with algebraic expressions the clear understanding of the definition of function is not essential; students respond to this latter form through certain stereotypical behaviours.

In respect with the two research questions, students' ideas are organised around two poles. The first is that of the conceptual understanding of function, which strongly connects with representations in the form of mapping diagrams and Cartesian graphs, and therefore has a higher level of success when dealing with most of the representations of functions. The other is the one dealing with function as a completely ambiguous relation, which connects with stereotypic forms of function that can be easily identified. Further research is essential in order to examine whether the formation of the above two poles may be modified through appropriate instruction.

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PRACTICAL CONSTRAINTS UPON TEACHER DEVELOPMENT IN PAKISTANI SCHOOLS

Dr. Razia Fakir Mohammad

Abstract

In this paper I discuss the impact that both conceptual and contextual problems have in inhibiting teachers' disposition towards capacity for development. These problems were highlighted from teachers' participation with a teacher educator in a collaborative culture of learning and within their schools' culture. They were challenged, supported and committed to teaching for achievement of their new aims deriving from an in-service course at a university in Pakistan. The teachers' capacity to learn was increased during the period of research; however, they needed support in dealing with issues for further enhancement of their teaching. The analysis of the teachers' transition from their routine teaching to new teaching revealed the teachers' needs as well as a gap between theory and practices in teacher education. I conclude the paper by suggesting to the community of teacher educators (including myself as a member of this community), that we should revisit our perspectives of teacher development at the university in the light of practical reality in a school context.

Introduction

This study contributes to an understanding of, and hence to improvement of teacher education on which the education of children in Pakistani schools subsequently depends. The guiding principles behind this research were that reflection and justification of self-actions would enable participants to understand the reality and difficulties of practice and their own contribution to achieve improvement in practice in a collaborative partnership (Wagner, 1997; Jaworski, 2000). The research was premised on the idea of shared ownership in order to support and examine teacher implementation of new ways of teaching resulting from their learning in a teacher education course. The findings suggest that during the limited period of this research, the participants were able to go through only the initial stages of the learning process. Nevertheless, I as a researcher came to realise that the teachers had started to adapt teaching strategies, and discuss issues in their teaching such as their misunderstanding of students' responses and their own understanding of mathematical concepts. By working very closely with the teachers, I was also able to understand some of the issues of implementation of teachers' new learning resulting from their university study.

Context of the Research

Three teachers, Naeem, Neha and Sahib participated in the research. They had resumed their teaching after attending an 8-week in-service course for teachers of mathematics at a university in Pakistan. The new way of teaching mathematics discussed in this course was based on a social constructivist perspective of learning, on the idea that learners are active creators of their knowledge and not passive recipients that a teacher can fill with knowledge. A teacher's primary responsibility is to assist in the learners' cognitive restructuring and conceptual reorganisation through providing opportunities for social interaction in mathematical tasks that encourages the discussion and negotiation of ideas (see Cobb, et al., 1991; Jaworski, 1994). My study was designed to follow up some of the teachers after the course and to support the teachers in developing their teaching according to their new aims.

All three teachers aimed to increase students' participation in their own learning and develop students' conceptual understanding of mathematics (adapted from their

learning at the university). This became evident when they accepted my invitation to participate in my research and expressed the following individual aims:

- Sahib's aim was to talk about the classroom issues and plan lessons according to new methods.
- Neha's aim was to plan and teach lessons according to the ways that she had learned in the university course.
- Naeem's aim was to discuss how to teach mathematics with reasoning.

Methodology

The nature of my research was reflective and participatory. I adopted interpretive research methods, collecting data by audio-recorded conversations in pre- and post-observation meetings; maintaining field notes during the teachers' participation in teaching or in their learning with the teacher educator along with the teachers' written comments (when provided) and my own reflective journal entries. The data were collected and analysed in the teachers' native language of Urdu. In my analysis, I studied each teacher across lessons and identified a range of issues in the teachers' learning. I checked that issues emerging across the three cases were indeed representative of the data as a whole. By listing all the examples that uncovered particular issues for each teacher, I was able to identify those that were common to the three teachers or distinct from each other.

Findings and Discussion

In this section I address issues germane to the thinking and practice of all three teachers that resulted from my cross-case analysis. However, due to the space limitation I discuss the examples from one teacher's practice according to new aim of teaching.

Teachers' Understanding of Students' Answers

The first part of my analysis uncovered teachers' difficulties both in addressing students' thinking processes and in helping students get the right answers through their own mathematical reasoning. The teachers' stated focus was to increase the students' participation in their learning but their practice did not provide evidence of their attempting to view students' solutions from the students' perspectives. Nor did the teachers deal with what seemed like errors and confusion in students' current understanding. The teachers did not seem to notice that students had developed different interpretations of what the teachers had been asking or presenting in the class. Nor did they clarify what was inappropriate in students' explanations and why. I observed that the children were left alone with their confusions and were without any clear justification for the correctness or incorrectness of their answers.

Below, I present part of the conversation between the teacher (Sahib) and student, in which I observed only a routine way of dealing with the students' answers.

1 T:	Somebody has thought of a number, multiplied it by three, subtracted one and got five. Tell me the number he has thought of. *The teacher also wrote on the board*, **x*3 – 1 = 5**
2 S:	Two
3 T:	How did you find it?

	The teacher called that student to the board and asked him to write his method.
	The student wrote on the board:
	2*3 =6 –1 = 5 [the student multiplied 2 by 3 first, getting 6, and then subtracted 1 to get 5]
	[The way it appears in writing is mathematically wrong (since 2 * 3 is not equal to 6-1); however, the student seemed clear in thinking while writing. He first multiplied and wrote the answer and then subtracted 1 from the product and got the result].
4 T:	How did you get 2?
5 T:	Good. [The teacher used this word often, I assumed that this was his expression to encourage students' participation] How did you find this?
	The student was silent. The teacher then asked other students to explain in words what
their	friend wrote on the board. There was no response. The teacher then told the student.
6 T:	First, you added one to five and you got six on the other side. Then you divided six by three to get two.

In the above example, the teacher explained the student's symbolic representation in a very different way to that used by the student (see line 6 in the data). The teacher's imposition of his own procedure and his rephrasing of the students' answers (after inviting the students to bring their own ideas) first encouraged and then discouraged participation. This led to confusion and sustained dependency on the teacher. This was evident in students' subsequent silence in the classroom in response to the way the teacher dealt with their explanations. The teacher's interpretation affected the student's level of confidence, because after that example none of the students offered their thinking process, either verbally or in writing. For example, Sahib then gave another equation, x * 4 – 3 = 5, and asked for the answer. One of the students said it was 2, but none of them then expressed a method to get the answer (either symbolically on the board, or verbally). In my analysis, the first student had his own way of thinking but the teacher ignored the student's way of thinking and did not confirm the student's method (see lines 2 to 4).

During our discussion, I asked the teacher about his different way of expressing what the student had written. The teacher reasoned that he wanted to teach a proper method. He also talked about the students' poor background of mathematics as a barrier in increasing their participation. In the teacher's opinion, it was very time-consuming to involve students and expect them to explain their thinking. He said that if he had taught the same lesson traditionally, he would have finished the entire exercise in the textbook.

Teachers' Mathematical Content Knowledge

The teachers' aim to teach mathematics with reasoning challenged their own understanding of mathematics. The problem of the teachers' limited conceptual understanding, their reliance on prescribed methods and particular answers, became evident when they came to express their mathematical point of view while planning, teaching and analysing lessons from beyond the textbook. All three teachers were, at times, unable to review, clarify and rationalize the mathematical assumptions behind the textbook exercises. The following example of Sahib's teaching 'division in algebra' illustrates the gap of teacher content knowledge.

Sahib began the lesson by testing the students' knowledge of basic algebra; for example, definitions of *variable, constant* etc and then he drew the following table and explained the rule of 'powers of two'.

2^4	2 x 2 x 2 x 2	16
2^3	2 x 2x 2	8
2^2	2 x 2	4
2^1	2 x 1	2
2^0		1
2^{-1}		$\frac{1}{2}$
2^{-2}		$\frac{1}{4}$

After that explanation, the teacher wrote the question $\frac{xy}{x}$. He solved the question in this way: $x^{1-1}y = y$. However, he did not provide any linkages between his explanation of the rule involving powers of two and his solution to the question. Then he gave another question and invited the students to solve this on the blackboard. The whole class sat listening to the teacher; none of them raised their hands. The teacher's intention seemed to be helped the students to generalise the rule of exponents from that example of 'powers of two', and to apply the rule in the presented task. However, he did not provide adequate explanations to support the students' ability to understand such questions. I observed silence in the class. The teacher himself did not seem to understand the barrier of his own limited knowledge impeding achievement of his aims of helping the students to understand the questions.

Imposed Identity

The shift in the teachers' goals following their learning at the university demanded that they use their intellectual capabilities in planning, teaching and evaluating their lessons, contrary to routine practice. However, the teachers appeared to be highly routine-bound. The new aim in their teaching was to enable the students to be independent through allowing them to solve problems in their own ways. In practice, however, the teachers dominated the discussion, thus limiting the students' participation. For example, in the lessons (discussed above) Sahib discouraged student participation in spite of his expectation that he would increase their participation. Although, the teacher's intention (of child-centred learning) was mentioned as an objective in his oral planning, in his practice he did not move from very traditional interaction. In fact, all three teachers interrupted and directed the students' thinking through their continuous comments and questions. It was difficult for them to reduce their own domination of the lessons, to stop telling the students what to do or to provide the students with the space to organize their thinking. The teachers' behaviour did not allow the students to step back from dependent modes of behaviour, despite the teachers' aims and explicit intentions to do so.

Their habit of working in a teacher–dominated culture seemed to create mental barriers to self-analysis for all three teachers. The teachers' analyses of a lesson focused mainly on what the student said, what the teacher wanted and what the wider social problems

were in relation to the achievements or failures of their new aims of teaching. They appeared unable to critique their own mathematics and mathematics teaching. For example, Sahib talked about students' dependency and lack of interest but he did not to realise that it was possibly his own authoritarian stance that maintained the students' dependency. My view is that the teachers were unaware of the complexity of practice in relation to their new aims. These teachers claimed to be willing to change their practice but were unable to cope with the challenges. Perhaps the teachers lacked understanding of the concept of improvement itself or did not realise the difficulty of introducing changes in their classroom without perceiving and challenging the complexity of their habitual constraints.

Time Consuming Approach

The teachers faced difficulties in achieving their new aims of teaching within available school and lesson time. Their new practice demanded quality time to comprehend and rationalize new aims and new practices; however, that time was out of the teachers' reach. The time these teachers contributed to the research partnership was their non-teaching time at school in which they had to fulfil regular requirements such as marking. They had replaced this routine work with discussion in relation to achieving new aims of teaching. However, the cost of such replacement was their own time at home. Despite their devotion, the time was still not sufficient for the teachers to satisfy the expectations of their new role. This resulted in additional pressure on the teacher to continue the lesson on the following day. For example, Sahib commented on his effort to increase students' participation:

> I cannot teach according to the new way; if I give them thinking time I would not be able to concentrate on written work. Tomorrow I have to continue this exercise, I cannot move to another before this.

Working Conditions

This section focuses upon the working conditions within which the teachers participated in their development of teaching at the schools. The teachers were under pressure of their annual appraisals, their completion of the textbook and students' examinations. These limitations affected the teachers' practice and confidence in tackling their new aims of teaching. The teachers' prior experience of their appraisal had minimized their capacities to improve teaching. For example, Sahib's reasons for ignoring the student's answer and imposing his rule (as discussed above) were his negative recall of prior experiences of evaluations by an inspector.

> I have to consider an observer's [inspector] thinking during my teaching; an observer could evaluate a teacher negatively when the students give answers that the teacher is supposed to tell them. I have mentioned this issue in my reflective journal also. If you [the teacher] ask a question and a child gives an answer then an observer thinks that the teacher has told him it before hand.

Elaborating his comments, Sahib said that in his prior experience an inspector of the school had misjudged his aims of the lesson when a student had provided an explanation, which was supposed to be given by the teacher as an introduction to that topic. The inspector evaluated the teacher as previously having taught that lesson. The inspector did not understand the capabilities of the child in thinking or appreciate the value of the teachers' questioning in the lesson. Sahib said that the presence of the

teacher educator in the class reminded him of the inspector's perspective, and, in consequence, limited his dealing with students' answers. Sahib expressed his concern also in his reflective journal while discussing the issue of students' equal participation in a lesson. He wrote:

> If the students do not give an answer then the school [inspector] thinks that they have not learned anything. If they give unexpected answers then the impression is that the teacher had taught the same concept before. The blame is always on a teacher.

All three teachers had pressure to complete and revise the textbook, so the students could memorize sufficiently and practice to pass their examinations. For example Sahib stated:

> I have to complete the syllabus before the final examination. … We check their memory and skills of drawing [geometrical shapes] in examination; conceptual clarification is not a basic requirement of the examination. If we 'check' [assess] their concepts, none of them will pass the examination.

The teachers also discussed the tensions and frustrations resulting from their low financial and social status in society. Their financial stress required the teachers to do more jobs besides teaching. These teachers asked questions about betterment of their financial status, workload and family responsibilities.

Implications and Recommendations

The above discussion has shown that the teachers' conceptual and contextual constraints restricted them in conceptualising the underlying assumptions of the philosophy of the teacher education course in the practicality of their new roles in teaching. They experienced difficulties making improvement within existing conceptual and contextual constraints although they wanted to adapt their practice according to new aims of teaching. The teachers were aware of some limitations but did not know how to deal with them.

My view is that by encouraging students to participate actively (contrary to a traditional mode of teaching) teachers effectively open up a possibility of learning with understanding. However, teachers' lack in making sense of students' responses and actually dealing with them may encourage teachers to sustain their prior identity wherein they give preference to their own knowledge and impose their own decisions. Thus, this pattern could sustain a cultural norm of 'underestimating students' strength'; that is, placing blame on elders is not acceptable in parts of Pakistani society where it is assumed that children have low potential for thinking, and wisdom occurs through age and experience. My own development as a learner and teacher also testifies strongly to this analysis. Further, teachers' attempts at achieving child-centred learning within the limitations of their own understanding of new practice may itself cause intellectual, emotional and affective hindrance of students' growth. If a teacher does not understand or deal with students' answers, what is the motivation for students to supply their own answers? In addition teachers cannot develop professionally with their limited content knowledge. Teachers need to enhance their mathematical understanding in order to understand what constitutes teaching of mathematics with reasoning (Ma, 1999). Limitations of mathematics content knowledge can be a big threat for teachers' confidence and desire for developing teaching.

In Pakistani schools, mistakes are generally not accepted because there is a focus on the product, on 'the what' instead of on the process and 'the why'. For example, in one case when a parent asked for clarification of the teacher's explanation (that was different from the textbook explanation), the teacher was threatened. The teacher reverted to the textbook and blamed the student's carelessness in listening to the teacher, because she wanted to avoid further complications and misjudgments. The teacher did not want to be dishonest but more important concerns were her job evaluation and her position at the school. Confessing a lack in knowledge is generally considered as a matter of shame and threat. This is highly embedded in the cultural norms. If teachers make efforts to improve their teaching, they may run a risk that their efficiency will be viewed negatively because it exposes their lack of knowledge and this will be seen as having a negative effect on students' learning outcomes.

The analysis suggests that in order to implement new methods of teaching teachers need time to plan lessons, and to consolidate planning so as to act accordingly in the class as well as to reflect on the outcomes of teaching,. However, time is a constraint in the school. Teachers could correct work and transmit knowledge from one class to another class, in the time available to them, but planning, teaching and learning according to new aims require more time. This leads us (a community of teacher educators) to think about ways to alter the 'time consuming approach' to a 'time reaching approach' in order to increase possibilities for child-centred learning in the real context of a school.

Moreover, due to an unsupportive school culture, routine teaching could be considered a secure, convenient and compensated option for teachers, because it protects their time, stress, position and promotion in the school, although it does not enhance their understanding of their professional development or contribute to students' understanding of concepts. From my analysis, questions emerge for the community of teacher educators: Can teachers achieve any improvement, if the culture works against the teachers' improvement? How can Pakistani teachers maximize their learning capacities if their self-esteem is low? What can the nature of teacher education be in these circumstances and within these limitations? How can we, as teacher educators, liberate teachers from the imposed constraints of schools in their contemplation of change?

Thus, due to the practical constraints teachers may put a layer of 'new practice' on top of their traditional practice in response to what they learn from in-service education but without any integration between these layers. This may prohibit them from acknowledging their inner resistance. This conflict might result in a tension of living between two practices, thereby extending the gap between theory and practice instead of closing it.

The following might assist in supporting teacher development:

- Teachers need help in enabling students to understand mathematics with reasoning if they want to promote their teaching practice. In addition teacher educators need to find ways of enabling teachers to conceptualise their work with pupils in the classroom, i.e., how to get right answers with an incorporation of students' mathematical reasoning and teachers' own standards.

- Teachers need to enhance their mathematical understanding in order to understand what constitutes teaching of mathematics with reasoning. Teacher educators need to have great sensitivity to, and understanding of, the consequences of the teachers' limited knowledge of students' learning as well as implementing the learning from a course. They need to relate the content knowledge the teachers have to teach to their students together with appropriate methods.
- Teacher educators need to address the problem of the length of teaching time required for a lesson and length of non-teaching time at the school as well as how to adjust new teaching in the available time in relation to introducing innovative ideas from the university.
- Teacher educators need to discuss ways to establish a learning environment in the school where teachers focus on students' learning and understanding together with fulfilling textbook requirements with limited resources and within the school expectations.

Change is utterly dependent on the needs of teachers, its compatibility with the reality of the school context and the provision of support. Preparing teachers for change without addressing their needs and providing ongoing support at their school would not allow teachers to acquire a breadth of improvement within their new practice. If teachers' needs and the requirements of a support system are ignored the tensions between theory and practice will continue. I conclude the paper with a comment from one of the teachers:

> If I move back to my previous style [of teaching] then there will have been some reasons and pressure. It will not only be my fault. We need to work as a group if we want improvement.

This statement suggests many questions from a teacher to a teacher educator; from a school to a university or from practice of routines to a theory of change.

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TOWARDS A DEFINITION OF FUNCTION

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*This paper points up, in the case of a particular class discussion, the crucial role that the **Trace** tool could play as potential semiotic mediator for the notion of function. In particular, the episode we are presenting here want to show how the idea of trajectory developed through a specific sequence of activities, carried out in Cabri and centered on the use of this tool can substantially contribute to building the meaning of function as a point by point correspondence. It also shows the conceptual difficulties attached to a complete construction of this meaning and how the role of the teacher is based on and complements the pragmatic experience of the students in Cabri.*

INTRODUCTION

Similarly to what happened for other basic mathematical notions, a formal definition of function, as correspondence between two sets, dates back to the beginning of the nineteenth century. Actually, within set theory, the definition of function as a particular triplet (**E**, **F**, **A**) in which **A** is a subset of **E**X**F**, is due to Bourbaki and it has been given in 1939 (Bourbaki 1939).

As shown by Malik (1980): "a deep gap separates early notions of function, based on an implicit sense of motion, and the modern definition of function, that is "algebraic" in spirit, appeals to discrete approach and lacks a feel for variable".

Nevertheless, it's interesting to remark that traces of the fertile nexus of this concept with the sense of motion can be still identified in the work of famous mathematicians that contributed to the elaboration of this modern definition. In fact, in 1837 Dirichlet writes: *"Soient a et b, deux nombres fixes et soit x, une grandeur variable, qui prend successivement toutes les valeurs comprises entre a et b. Si à chaque x correspond un y fini unique de façon que, quand x parcourt continûment l'intervalle entre a et b, y=(x) varie aussi progressivement, alors y est dite fonction continue de x sur cette intervalle. Pour celà, il n'est pas du tout obligatoire que y, sur tout l'intervalle, dépend de x par une seule et même loi, ni qu'elle soit représentable par une relation exprimée à l'aide d'opérations mathématique.[1]"* (Youschkevitch 1976). As Youschkevitch underlies, the general characteristic of this definition of continuous function and its possibility to be directly generalized to the discontinuous one is evident; nevertheless a dynamical point of view is still present and Dirichlet found

[1] "Let *a* and *b* be two fixed numbers et let *x* be a variable quantity that takes successively all the values between *a* and *b*. If for each given *x*, a unique finite *y* corresponds to it in the way that, when *x* moves continuously along the interval between *a* and *b*, *y*=(*x*) varies progressively too, then *y* is said to be the continuous function of *x* over this interval. For this, it is not obligatory at all either that *y*, on all over the interval, would depend on *x* according to the same unique law, or that it would be represented by a relation expressed with the help of some mathematical operations" (translated by the authors)

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the necessity to add a geometrical explanation to this definition.

On the contrary, the modern definition of function refers to a static notion that has lost every relation with the primitive dynamic intuition tightly tied to time and movement.

This research report presents a meaningful episode of a larger teaching and learning project, the first part of which has already been presented in PME27 (Mariotti et al. 2003). There were four classes involved in the project, two in France and two in Italy; the students were 15-16 year old with a major in scientific studies. In the previous research report (Mariotti et al. 2003) we have shown a particular aspect of the role of semiotic mediator played by some Cabri tools, in constructing a net of interrelated and indispensable meanings for the notion of function. In particular, both the asymmetrical nature of the independent versus the dependent variable and the twofold conception of trajectory (as both a "sequence of position of a moving point" and "a globally perceived geometrical object") were identifiable in the analysis of pupils' productions**.** Such meanings are crucial components for grounding the notion of function as a co-variation (between two variables, one depending on the other, and between two sets, the domain and the image) and they clearly emerged in relation to the internalization of the ***Dragging*** and ***Trace tools.***

Without solution of continuity to what we have already presented, the episode we are going to analyze here, enables to highlight an other aspect of the potentialities of the ***Trace tool*** as semiotic mediator. In particular, it shows the contribution of this tool in the emergence of the idea of function as point by point correspondence, by its simply evocation, in the case of a classroom discussion. This episode shows, also, how the achievement of this mathematical definition is difficult for the students and how the teacher manages to exploit such tools potentialities in order to attain this objective. Finally, this episode points up the role played by the problem of "defining two equal functions" in the construction of the meaning and of the definition of the function itself.

THE EXPERIMENTAL CONTEXT

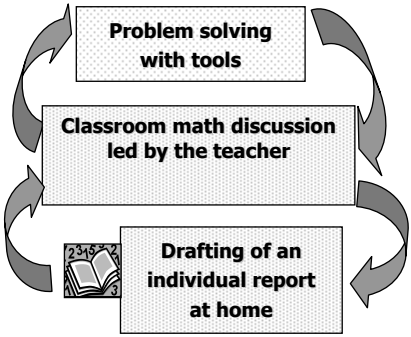
The sequence of activities, the episode presented here is part of, is based on four fundamental hypotheses:

1. One crucial aspect of the notion of function is the idea of variation or more precisely of co-variation, that is to say a relation between two variations one depending on the other one.
2. The primitive metaphor of co-variation is motion, that is to say the change of space according to the change of time.
3. A DGS environment, such as Cabri-géomètre, can provide a semantic domain of space and time within which variation can be experienced as motion.
4. According to the Vygotskian theoretical perspective of the semiotic mediation (Vygotsky, 1978, Mariotti, 2002), the computational tools and objects students

interact with, can be thought as signs referring to this notion of function as co-variation and, as such, they may become tools of "semiotic mediation", specifically implied by the teacher in class activities.

As a consequence of these theoretical assumptions, the general structure of the experimental sequence consists of four stages:

- At the beginning, students are faced with tasks to be carried out with Cabri tools;
- secondly, the various solutions are discussed collectively under the guidance of the teacher. These collective discussions play an essential part in the teaching and learning process. They are real "mathematical discussions" in the sense that their main characteristic is the cognitive dialectics between different personal senses and the general meaning which is introduced and promoted by the teacher (Bartolini Bussi 1998).



- Thirdly, it's required to the students to write at home an individual report specifying, on the one hand, what one has experienced and understood, and, on the other hand, doubts and questions arisen. This third stage is important because it constitutes a first externalization of internalized meanings and enables a reflective feedback on the solving task activity with tools.
- In a fourth time, students are asked to discuss about their productions. This phase is aimed at pursuing the processes of internalization and social inter-subjective construction of meanings.

THE EPISODE

A definition of "equal functions"

In the first part of the sequence, the activities with the Cabri-tools made the students perceive the difference between points that can be dragged directly, by taking them with the mouse, and points that can be moved only indirectly, by dragging those that these latter points depend on. This has become a reference situation, where a system of signs has been established, on the basis of which the meaning of variable has been introduced by the teacher. The points that can be moved directly correspond to the independent variables, whilst the points that can be moved only indirectly correspond to the dependent variables. Similarly and accordingly to our main fourth hypothesis, the use of the ***Trace tool*** contributed to the emergence of the twofold meaning of trajectory. In fact, both the conception of trajectory as a "globally perceived object" and as "an ordered sequence of position of a moving point" can be found in pupils'

formulations and individual reports at home. (Mariotti, et al. 2003; Falcade 2003).

After the first phase of activities, a collective discussion was carried out, with the explicit aim of elaborating a definition of function. The discussion was articulated into two parts and took place during three lessons (lasting approximately 5 hours), with a twofold aim (it is possible to recognize a cognitive and a meta-cognitive level) corresponding to:

- clarifying and systematizing the ideas emerged during the previous activities.
- expressing these ideas into a 'mathematical statement', i. e. the definition of function.

At the very beginning pupils were asked to characterize a function. Different elements in play are highlighted by the students: the (independent and dependent) variables, the range domain, the image. Both the pupils and the teacher refer to Cabri tools and Cabri phenomena, as experienced during the first activities.

The difficulties arising in entering the mathematical world make the role of the teacher become relevant; the teacher has the difficult task of mediating between culture and pupils, between mathematics, as a product of human activities, and pupils' learning. Thus when the crucial point arises and the pupils realize that characterizing a function implies determining when two functions can be said to be "equal", the teacher shifted the focus of the discussion and asked the student to try and give a "definition of equal functions". A first attempt of definition simply stated: « Two functions are equal if they have the same range domain and the same image ».

At this point it is impossible to say whether, when the pupils speak about range and image, they are thinking globally or punctually.

The teacher asked the students to go back to Cabri and to look for different examples that can corroborate or invalidate their first conjectured definition. After this working moment in pairs, pupils were asked to express the new ideas arisen from their activity in Cabri. The following definitions about "two equal functions" were proposed.

Andrea – Alessandro: "Two functions are equal if they have the same range domain and the same image for all the domains subsets of the original domain which define the functions"
Gioia – Federica: "Two functions are equal if they have the same number of variables, the same range domain, and the same procedure (in the construction of the macro)".
Marco- Gabriele: "Two functions are equal when they have the same image and (when) the same range domain is fixed (for both).
Tiziano – Sebastiano: "In our opinion two functions are equal if having the same range domain and the same definition procedure they have the same image. If either the domain, or the definition procedure, or the image are not equal, neither the functions are equal."

Apart from that of Gabriele & Marco, who do not take into account the procedure, all the other definitions do consider the main elements in play: the range domain, the procedure and the image. For the majority of the students, the attention is focused on the difference between the procedures, but, surprisingly, the definition of Andrea &

Alessandro presents a characterization in which the domain is thought in terms of subsets. It's a static definition that shows no traces of variations and uses a quantifier ("for all"). This way of thinking may appear quite strange, if one does not take into account the very peculiar experience that pupils had in the previous activities and the relation built between the idea of trajectory and that of image: in the previous work, in order to compare two different functions pupils have compared two procedures on the same range domain observing that each procedure produced a different 'trace – trajectory' (image). Nevertheless, the link between what it has been done in Cabri and Andrea's and Alessandro's formulation is not immediate at all, we can suppose that process of internalization of this tool, which has transformed the Dragging in Cabri into such a static formulation ("for all the domain subsets") has been quite important.

Let see how the way of thinking was shared in the class and evolved.

A "reliable" definition hard to be accepted

The discussion starts when pupils are asked to compare the different definitions they have produced.

1. Ins: we must find an agreement on a definition, which can be one of these, or an improvement of one of these, or the fusion of these … We must decide.
2. Andrea : According to me, Gabriele's and Marco's definition is wrong.
3. Ins: So, Andrea, according to you, Gabriele's and Marco's definition is wrong. Let's read it again (she reads again) "two functions are equal when they have the same image and (when) the same range domain is fixed for both".
4. Andrea : Because to get to the same image, someone could pass through… we could have several journeys; in fact, if there were a subset of the domain… we can't say that the functions are…
5. Ins :… Tiziano, could you try to explain it?
6. Tiziano: Yesterday, we saw that we can, by doing the same domain, we can create the same image and this, with different functions (procedures).

The teacher redirects the discussion on the comparison between the definition of Andrea and Alessandro and those referring to the procedures.

44. I Ins: Let's read the text. You say that if they have the same domain and the same image for each subset of the domain…
45. Tiziano: But, here it's like to have the same procedure.
46. Ins: Hum, and why it's like to have the same procedure?
47. Several voices: …Because…
48. Gabriele: …As we go further, the subsets of the domain and vice versa…
49. Ins: Do you agree, Andrea?
50. Gioia: The domain is the plane, then you have the straight line, then a segment…
51. Ins: What are these?
52. Andrea: The domain can be whatever.

53. Gioia: They are subsets.
54. Ins: And then, the procedure, what does it do? That is to say, I…. Where does it start from?
55. Andrea The domain can be one point too… if we want!
56. Ins: The subset of the domain can be one point too. Oh!
57. Andrea: For whatever point, we get the same point of the image.
58. Ins: And this give the idea to say that…
59. Gioia: I'm doing the same procedure.
60. Andrea (together with Gioia) I'm doing the same procedure
61. Ins: I'm doing the same procedure. Therefore, for whatever point of what?
62. Andrea: For each point of the domain we have the same… as the result of the function, the same point of the image.
63. Ins: Do you agree? (referring to Tiziano)
64. Perplexed silences
65. The teacher writes at the blackboard and reads: "For each point of the domain, we have as the result of the function, the same point as the image".

It is possible to observe the emergence of the idea of coincidence point by point, as it is originated by the coincidence trajectory by trajectory, passing to the limit situation when the subset is a point. This is the case for Andrea, in which the process of internalization of the ***Trace*** tool turns out to be quite substantial. The development of this definition is achieved by thinking aloud and other pupils seem to participate to its elaboration (see in particular the interventions 50 – 59, when Gioia completes the sentence of Andrea or answers instead of him), but we can't be sure about them.

At the beginning, pupils seem clearly to accept that "to have the same domain and the same image for each subset of the domain it's like to have the same domain and the same procedure (lines 45, 48, 50). This is probably due to the fact they have already experienced in Cabri different phenomena according to which, Andrea's and Alessandro's definition appears sensible. Their agreement with Andrea is based on their actions with the tools; it is not theoretical at all.

During the second part of the dialogue, involving mainly the teacher and Andrea, but also Goia, the definition emerges in a quite "logical" and reliable way. Nevertheless, at the end, when the conclusion is written on the blackboard and read by the teacher, it becomes difficult, for the other students to accept it. Actually, this corresponds to deny the key role of the procedure in the definition of a function. It corresponds also to overcome the conceptual move from an experience based definition, tightly tied with Cabri activities, to a purely mathematical definition. And, indeed, a further discussion was, needed to reach the acceptance of confronting point by point a function. The role of the teacher is crucial in helping students to face this move. Indeed, her role is determinant all aver the discussion. At the beginning (line 1) she states the didactical contract within which the discussions should be developed. In different occasions (lines 44, 46, 51, 54, 58, 61), she intervenes or poses very specific

questions, in order to redirect the discussion and focuses on the main mathematical points. In particular, at line 56, she repeats, with emphasis, Andrea's statement. She is aware about the important mathematical implications of Andrea's observation and pushes further the discussion in this direction. In other moments (lines 5, 49, 61) she tries to involve into the discussion students that seems not to participate to it. In general, what she tries to do is to orchestrate all the interventions in order to obtain that certain mathematical meanings emerge from particular students and then are discussed (and possibly shared) by all the other ones.

CONCLUSIONS

The excerpt, we have just presented, shows, in the case of a class discussion, a particular way the ***Trace*** tool can function as potential semiotic mediator. In fact, the idea of trajectory, as it emerges from the activities carried out in Cabri and centered on the use of this tool, substantially contributes, at least for Andrea, to building the meaning of function as a point by point correspondence. The same idea leads the other students to conceive that "to have the same domain and the same image for each subset of the domain it's like to have the same domain and the same procedure". The definition of function as correspondence is not far from there but it isn't immediate at all. Indeed, the procedural aspect seems to remain dominant for the other students and the correspondence between the two points, far from being arbitrary, must be related to a well stated procedure. The ultimate perplexity to accept Andrea's definition of "equal functions" shows also the difficulty to shift to a formulation that is theoretical and completely detached from the sensible experience in which it has been originated

Maybe the activities developed in the Cabri environment could have even reinforced a natural procedural tendency. But, on the other hand, within Cabri, the available tools (***Dragging***, ***Trace, Macro***,…) and the particular signs (segments, rays, Cabri figures representing the range domain, on which the independent variable varies, or the image, on which the dependent variable varies) offer a common semiotic system that the pupils and the teacher can elaborate. The Cabri tools and the related signs allow a discourse on them and on their behavior which gives a fundamental contribution to the construction of the net of interconnected meanings concerning the notion of function.

This excerpt shows also the importance of the teacher's role. On the one hand she has to organize a sequence of tasks involving tools to activate and support the process of internalization. On the other hand, she has to orchestrate the discussions in order to guide this process towards the construction, necessarily inter-subjective, of a certain specific mathematical meaning which may be, sometimes, quite different from the students' personal meaning. For this reason, in some cases, and this is the case, she has even to induce certain conceptual moves in order to help student to completely accomplish this processes.

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"WHY DOESN'T IT START FROM THE ORIGIN?": HEARING THE COGNITIVE VOICE OF SIGNS

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Starting from a situated cognition perspective, this paper reports on the activity of 9th grade students who are interpreting the shape of a graph arising from the motion of a bouncing ball. In an unfamiliar context, informed by previous knowledge of similar experiments, the obstacle of understanding why the graph does not start from the origin is overcome through an interplay between different signs.

THE IDEA OF A *SITUATED COGNITION*

> "When we deliberately and effortfully think, presumably muscles come into play. Primary among these are the speech muscles; for thought, as John B. Watson claimed, is primarily incipient speech. Thinking aloud is just uninhibited thinking. Other muscles enter the thought process too, as Watson appreciated, notably in the case of the artist or acrobat who plans his moves with incipient rehearsals of muscular involvement, or the engineer, who simulates in his muscles the lay of the land or the distribution of stresses in what he means to build. The artist, engineer, and acrobat are poor at putting their thoughts into words, for they were thinking with nonverbal muscles" (Quine, p.88).

The quotation above sheds light on the important role of body in the act of thinking; moreover, it stresses that this is not merely a matter of thought in itself. I argue that such a viewpoint holds just as for an *initial* phase of activation of thinking, as for the *constitutive* phase of its development; since this is true for all the human activities of thinking, it is significant for learning in particular. We have to bear in mind all the "ingredients" that constitute thought such as imagination, perception, motor-sensory receptivity, muscular activity, and brain information processing. Besides, thought is also matter of culture in a wide sense: learners' expectations, motivations, tasks, goals, previous knowledge, etc. Although each of those aspects is quite meaningful in its own right, none of them in isolation can fully tell the story of a student conceptualisation. Therefore it is necessary to speak not simply of students' cognition, but rather of what I call *cognition in context* or *situated cognition* (for a study within this perspective, see Watson, 1998). Since the reference discipline is Mathematics Education, *situated mathematical cognition* will be the general subject matter of my research.

SHAPING A CONSISTENT THEORETICAL FRAMEWORK

Recent developments and ongoing studies related to the psychology of mathematics highlighted biological and neurological constraints, cognitive mechanisms, and cultural roots affecting mathematical knowledge (see e.g. Butterworth, 1999; Dehaene, 1997; Houdé & Tzourio-Mazoyer, 2003; Lakoff & Nùñez, 2000). At

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present these new trends are not to be ignored because they clarify the central issue of the primary sources of mathematical understanding. More and more attention is being drawn to the role of perceptuo-motor activity (i.e., bodily actions, gestures, manipulation of materials or artefacts, acts of drawing, and so forth) in the learning of mathematics:

> "While modulated by shifts of attention, awareness, and emotional states, understanding and thinking are perceptuo-motor activities; furthermore, these activities are bodily distributed across different areas of perception and motor action based in part, on how we have learned and used the subject itself. […] the understanding of a mathematical concept rather than having a definitional essence, spans diverse perceptuo-motor activities which become more or less significant depending on the circumstances" (Nemirovsky, 2003, p.108).

This claim meets the idea of situated cognition. Furthermore, these perceptuo-motor activities are consonant with the semiotic "key elements in the organization of mental processes as they are used to reflect and objectify ideas in the course of the individuals' activities" (Radford, 2003, p.125): words, gestures, artefacts, drawings, and so on. As far as my research is concerned, the analysis has been focused on students' gesturing and linguistic productions, as observable constitutive components of the mental activity, by following the belief, to use famous words, that: "gestures, together with language, *help constitute thought*" (McNeill, 1992, p.245; emphasis in the original). My sense is that until recent years, gestures and language have often been taken into account in fragmented fashion, each in its own way, forgetting the richness of an interplay among them or between all the semiotic resources; but lately strong evidence has been published for their significance in constructing meanings in context, even in interaction with technology (see e.g. Alibali *et al*., 2000; Arzarello & Robutti, forthcoming; Nemirovsky *et al*., 1998).

A search for coherence: more signs

Among the contemporary cognitive oriented approaches, the theory of *embodied cognition* (Lakoff & Nùñez, 2000) has been for me the most fascinating one in that it investigates where mathematical concepts come from and examines the ways in which the embodied mind brings mathematics into being. Early interests were on studying the role of *conceptual metaphors* (the "fundamental cognitive mechanisms which project the inferential structure of a *source domain* onto a *target domain*"; Nùñez, 2000, p.9; emphasis in the original) in relation to mathematics learning. Later, metaphors came to be seen as vehicles of knowledge from the perspective of a social construction of meanings (Ferrara, 2003). On the other hand, we have analysed the mediation-role of technology in activities in which 14 years old students were asked to interpret on a calculator some graphs obtained from physical motion experiments in front of a sensor (Ferrara & Robutti, 2002). But soon, the question of a relationship between metaphorical thinking and the use of technology arose. Gestures and perceptuo-motor activities cannot be forgot nor understood without the means and

tools that they put in motion. Many more *signs* need to be looked at to have a framework *consistent* with the idea of a situated cognition.

Signs and symbols are used in the literature sometimes with different meanings, and sometimes as synonymous. It is then important to make explicit how we use such terms in this study. With *signs*, I refer to what Radford (2002) calls "*semiotic means of objectification* – e.g. objects, artifacts, linguistic devices and signs that are intentionally used by individuals in social processes of meaning production in order to achieve a stable form of awareness, to make apparent their intentions and to carry out their actions" (p.15). Gestures will be part of signs, together with particular words, conceptual metaphors and blends, results of perceptuo-motor activities, keys on the technological artefacts, mathematical and general objects (like graphs, number tables, bouncing balls, etc.). Instead, with *symbol* I mean *the final widest and highest stage in the life of a sign*: that specific kind of sign embedding a conflation of the signified and the sign, in the present case a *fusion* (see Nemirovsky *et al*., 1998) of the physical phenomenon of motion and its mathematical graphical representation.

Within this perspective, I will discuss the mathematical cognition of some students (9th grade) while striving to make sense of a position-time graph resulting from a bouncing ball. Roughly speaking, the paper arises from an attempt of *hearing the cognitive voice of signs (gestures, speech, artefacts, metaphors and so on) in such a situation*, trying to survey their role and interplay in making something non-palpable and unperceivable for students at the beginning (the shape of the motion graph) palpable and perceivable at the end.

THE CONTEXT

The experiment: methodology. The data comes from a long-term teaching experiment carried out a couple of years ago and part of a research project still in progress. The core activities involved grade 9 students in approaching the concept of function as model of a physical movement described through graphing (Ferrara & Robutti, 2002; Arzarello & Robutti, forthcoming). Two teachers, one of mathematics and one of physics, were active in the classroom and collaborated in designing the experiment sequence. In each activity the students worked in small groups (three to four people), then participated in a class discussion led by a researcher present in the classroom, and aimed at sharing and comparing the different solutions. We collected data for the analyses in the form of students' written notes, worksheets completed by the groups of students, and transcriptions from video-recordings of both, group and class discussions.

The activity. The activity this paper reports on is the fourth of a series. Their focus was on the construction of a model starting from physical motion (a second phase was centred on the inverse passage: from models to motions). In previous sessions, the students were asked to move in front of a CBR (a motion detector) to perform different kinds of motion: uniform (back and forth), accelerated and periodic. A red line on the floor marked the point where they had to stop or change direction. The

last session differed from these in that the students used a toy object: a bouncing ball that groups dropped under the motion sensor. As the ball bounces, the students observed the position-time graph built in real time on the screen of a symbolic-graphic calculator (TI-92 Plus) linked to the sonar. After gathering data (stored in calculator memory) we provided each group with a worksheet encompassing three steps: first, the students explained in natural language the motion of the ball and the graph (e.g. "Describe the kind of motion the ball made", "Describe how space changes with respect to time (increases, decreases, etc.)"). Second, they were asked to interpret *qualitatively* the shape of the graph (e.g. "Analyse the graph. Is it like a straight line? Is it like a curve? Does the curve increase? Does the curve decrease?"). Finally, they had to interpret *quantitatively* the graph by calculating the slope of the curve at different points (e.g. "Consider the ratio $m = (s_2 - s_1)/(t_2 - t_1)$ and use it to describe mathematically the graph of your motion"[1]).

At the time of the experiment, in the first months of the school year and of high school, the students had not developed formalised knowledge in terms of graphs or functions. From an instructional view, the activities were designed to let the students pass gradually from an intuitive stage, starting from verbalising the experiment through natural language, to a more *conceptual* one. Because of space constraints a brief excerpt from the activity of one small group of students will be considered. Despite the briefness, it is significant for the different signs coming into play; I will strive to hear and understand through the analysis what their voice is saying.

DISCUSSION

The CBR was raised around 2 *m* from the ground and the ball was dropped under it. Only the first part of the graph (Figure 1) until 6 *s,* during which the bounces of the ball can easily be noted, is of interest; after 6 *s* the ball drifted away from the sensor range. The students are discussing the questions about the shape of the graph: Is it like a line? Is it like a curve? Does the curve increase? Does the curve decrease?".

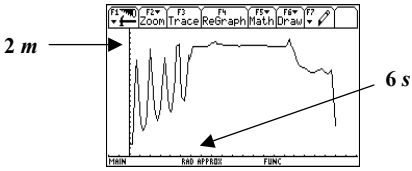

Figure 1

Soon a problem arises for Fabio and Giulia: the graph *does not* start from the origin. The *schema* (Radford, forthcoming) they have in mind differs from what the calculator screen shows. The previous artefact-mediated experiments, in particular the periodic motion (see the pronoun "we", #127), condition the students' expectations for the graph[2]. The schema needs to be re-thought of. But: *how?*

122. Giulia: *Why doesn't it start from the origin?*

123. Fabio: *When it arrives to this point here* [he is placing the cursor on an inferior extremity of the graph] *it is when the ball…*
124. Giulia: *Why doesn't it arrive on the horizontal axis?*
125. Fabio: *Here* [he is again placing the cursor on an inferior extremity of the graph] *it is as the ball, when it is near the CBR*
126. Filippo: *Yeah*
127. Giulia: *Hum, because we…*
128. Fabio: *Here it is when it* [the ball] *is near the CBR*
129. Giulia: *No*
130. Fabio: *Instead here* [he is moving the cursor on a superior extremity] *it is when…*
131. Giulia: *Here* [she is pointing to one of the superior extremity] *it is when it* [the ball] *is on the ground*
132. Fabio: *Here, when it is on the ground, just this point* [he is placing the cursor on the same superior extremity]
133. Giulia: *Yeah*
134. Fabio: *When it is on the ground* [he is moving the cursor on the next inferior extremity]
135. Giulia: *Whereas there, it is when it is near the CBR…*
136. Fabio: *And when it is near, when it approaches the CBR, the maximum point in which it approaches the CBR is here* [he is pointing to the inferior extremity of the graph where the cursor is located]

Not only the graph does not start from the origin but, in addition, it does not arrive on the horizontal axis (the physical ground, the floor, does not correspond to the 'mathematical ground'!). This requires a reflection both on the spatial origin of motion and on the position of the sonar (raised vertically, with the ball falling along a vertical trajectory). The point of view needs to be shifted to accommodate the ball as subject of motion. The passage happens with the help of a specific sign: the *Trace* key activated on the calculator displays a moving cursor on the graph, with the numerical coordinates corresponding to the point the cursor is at. Such a sign prompts a first exploration of the graph, rich of *deictic* and *locative* words (Radford, forthcoming): "here", "there", "this point"; "near the CBR", "on the ground". Usually these words are matched with physical body gestures; indeed, in the present case the *cursor* acquires the mediation function of an *index*, in place of usual *deictic/indexical* gestures (#123, 125, 130, 132, 134). Technology is working *with* the students, not *for* them: the instrument provides them with an inherent function (the tracing function) used through a physical tool (the cursor) to make accessible the motion experiment in the graph. The students already think of the graph in terms of motion (the pronoun "it" is used to speak of the curve 'starting', 'arriving', etc.); however, their interpretation remains at a *local* level, in terms of different spatial positions of the ball during its movement (this is an *operational* way to see the graph), as pointed out by the *timed* sentences (observe the pervasive presence of the temporal adverb "when"). The beginning of a more global conceptualisation of the graph is marked by their discovery that the maximum positions reached by the ball in motion correspond

to the inferior extremities on the graph. A step more is made in #136, where a gesture substitutes the cursor.

137. Giulia: *Well, a curved line represents the graph…*
138. Fabio: *A curved line*
139. Giulia: *A curve that doesn't start from the origin*
140. Fabio: *Wait, say, let's see, hum…*
141. Giulia: *It is a curved line that has always the same… maximum point*
142. Fabio: *That… when…*
143. Giulia: *…but that varies as minimum point*
144. Fabio: *When the curve goes up, when it goes up* [he is sketching a parabolic slope in the air with his right hand: his index finger, pointing to imaginary physical locations in front of his body, follows this going up trajectory, from the bottom-left to the top-right]*, it indicates that the ball goes down towards the ground* [he is lowering his hand in a vertical direction, miming the ball motion while falling]*,* [he fast raises his hand] *when it* [the curve] *goes down* [he is lowering his raised hand, reproducing with the index finger the previous slope but in the opposite versus]*, the ball* [he is raising his hand again in a vertical direction, miming the ball motion while bouncing up]...
145. Giulia: *It indicates that the ball goes up*

A second interpretation of the shape of the graph in terms of maximum and minimum points begins[3]. This is a *natural* resource for the students, so that they seem to have *embodied* the extreme spatial positions of the ball on the extremities of the graph (#141, 143). The students are now at a *global* level, in which the graph is considered in a *structural* way. Let us focus on the last two lines: Fabio shows of *knowing* and *acting* at once (#144): he knows the graph in terms of the motion of the ball, and he acts to reproduce both the shape of the curve and the motion itself. The schema has definitely been re-thought of: Fabio is able to enact the whole experiment and to *recognize* it *in* the graph, and Giulia shares this knowledge (#145). Graph and motion are not longer distinguished; in contrast they are *fused* together as marked by the use of verbs of motion, "to go up" and "to go down", for talking about the graph ("the curve", or "it" in the sentences) too. This process may be interpreted as the construction of a *blended space* (Lakoff & Nùñez, 2000), in which the features of the two domains of the *Graph is a Moving Object metaphor* (see Ferrara, 2003) are merged in a new unique sign: the motion-graph. On the other hand, this cognitive behaviour is also embedded in a convergence of Fabio's *iconic* gestures (Mc Neill, 1992), which are of two kinds. Informed by current trends in the study of gestures (see e.g. Alibali *et al*., 2000; Arzarello & Robutti, forthcoming; Edwards, 2003), I refer to them as follows: when Fabio's hand traces the shape of a piece of the curve, the gesture is *iconic-representational* (Arzarello & Robutti, forthcoming), standing for the graphical representation; when its movement enacts the motion of the ball, it is *iconic-physical* (Edwards, 2003), since it represents the physical phenomenon. The graph becomes something to think with, the motion something to enact. But, there is more information here: the transition from the operational (local) to the structural (global) mode of thinking of the graph mirrors the occurrence of a process of

reification, which, through the birth of the Graph is a Moving Object metaphor, marks the beginning of the conceptualisation (for a study about the role of metaphors in constructing new concepts, see Sfard, 1994). As a consequence, rather than being simply the result of a conflation of the two existing domains of motion (the physical phenomenon) and graph (the abstract sign), *the blend arising from the metaphor is what brings the abstract graph-sign into existence as a symbol*. Similarly, the two Fabio's gestures are so coordinated with speech and among them and iterated during the explanation that, recalling the distinction by Peirce (1955) between three different kinds of signs (index, icon and symbol), I think it is possible to speak of them as gestures with a *symbolic* characterisation.

FINAL REMARKS

The analysis reported in this paper points out the need of further investigations: reification, blends, metaphors, knowledge objectification, special kinds of gestures or actions seem to be due to a *basic and inherent cognitive mechanism* activated in constructing a meaning, in grasping a concept, in conceiving a sign as symbol. On the other hand, reasoning in abstract terms or on abstract entities (such as mathematical objects, e.g. a graph) is a complex activity that requires drawing attention to the deep network of interactions of the students with the environment they are acting upon. Although it is not yet clear the fundamental mechanism entering the scene (and this remains an open problem), support may come from our neuro-biological structure. Recent neuroscience studies, for example, shed light on the representational dynamic of the brain as *non*-symbolic but as a type of self-organisation, in which body action plays a crucial role (Gallese, 2003). Results of this kind are to be regarded with a special eye, because they may provide us with productive answers.

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1. t_1 and t_2 are two subsequent time data, s_1 and s_2 are the two corresponding position data. A table on the calculator provides the students with the numerical values.
2. In previous motions, the graph starts from the origin: in fact, the student moving approximately begins his/her run at time t=0 where the CBR can gathers data (0.5 m). And at that time he/she has not yet covered any space.
3. As a consequence of *pixel definition* of the screen, there is not a perfect match between the plane locations of the cursor on the inferior/superior extremities and the maximum/minimum points. But this is not a problem, because a qualitative interpretation is enough for the students.

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MATEMATICAL LANGUAGE AND ADVANCED MATHEMATICS LEARNING

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This paper is concerned with the role of language in mathematics learning at college level. Its main aim is to provide a perspective on mathematical language appropriate to effectively interpret students' linguistic behaviors in mathematics and to suggest new teaching ideas. Examples are given to show that the explanation of students' behaviors requires to take into account the role of context. Some ideas from functional linguistics are outlined and some features of the texts usually produced by students are discussed and compared to the corresponding features of standard mathematical texts. Some teaching implications are discussed as well.

INTRODUCTION

The role of language in mathematics learning is a critical topic, and it is usually dealt with from a variety of theoretical perspectives. A controversial issue is the relationships between communication processes and the development of thinking. In the opinion of some researchers[1] thinking and communication are closely linked, whereas others[2] regard them as quite independent processes. The language of mathematics itself is interpreted from a variety of perspectives. In the opinion of a good share of mathematicians the specific features of mathematical language chiefly reside in mathematical formalism. On the other hand, verbal language is widely used in mathematical activities (including research), and language-related troubles are not confined to the symbolic component at all. I presume that most mathematics educators, no matter the theoretical frame they adopt, would agree that linguistic problems may undermine any further intervention, for students might misunderstand what they are told or they read, or be unable to express what they mean. It should be widely acknowledged too that this issue grows even more important if groups of language minority students are involved[3]. Moreover, if one assumes too that *"learning mathematics may now be defined as an initiation to mathematical discourse, ..."*[4], and languages are regarded not as carriers of pre-existing meanings, but as builders of the meanings themselves, then the linguistic means adopted in communicating mathematics are crucial also in the development of mathematical thinking. So, poor linguistic resources would produce poor development of thinking.

The main aim of this paper is to provide a perspective on mathematical language appropriate to effectively interpret students' linguistic behaviors in mathematics and

[1] For example, Sfard (2001). Also Duval (1995) underlines the cognitive functions of languages in mathematics.
[2] For example, Dubinsky (2000).
[3] This topic is widely discussed in the book edited by Cocking & Mestre (1988).
[4] Sfard (2001, p.28)

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to suggest new teaching ideas. To achieve this goal some ideas borrowed from pragmatics (which is the subfield of linguistics dealing with the interplay between text and context) and functional linguistics (which is a theoretical stance within the field of pragmatics) are introduced. The colloquial way of using language (which is often the one adopted by students) is compared to the mathematical one through examples. The application of ideas from functional linguistics to mathematics has been carried out by a number of researchers such as Pimm (1987), Morgan (1996, 1998), Burton & Morgan (2000). Also Sfard's focal analysis (2000) might be related to standard topics of functional linguistics. Ferrari (2001, 2002) used the same ideas to interpret some empirical findings. This paper focuses on the theoretical aspects.

MATHEMATICAL LANGUAGE AND ITS USE

Through the paper I mostly refer to Italian Science freshman students and their learning problems in mathematics. At college level, students' troubles are customarily ascribed to the lack of specific contents in their high school curricula. On the contrary, my claim is that students' competence in ordinary language and in the specific languages used in mathematics are other sources of trouble.

To point out some aspects of this topic, I give a couple of examples. The following problem has been given to a wide range of samples, from grade 7 to college.[5]

Example 1

Link each sentence on the left to the sentence or the sentences on the right with the same meaning, if any.

a) Not all the workers of the factory are Italian.	a') All the workers of the factory are foreigners
b) No worker of the factory is Italian	b') Some workers of the factory are Italian.
c) Not all the workers of the factory are not Italian.	c') All the workers of the factory are Italian
	d') Some workers of the factory are foreigners

In all the samples (including college students) although most of the subjects properly treated sentence b), a good share connected a) to both b') and d'), and the same happened for c). The more suitable treatment (from the mathematical standpoint) of sentence b) compared to sentences a) and c) is a common feature of all the samples.

Sentence a') is equivalent to b) from the viewpoint of both everyday-life and mathematics. As regards sentences a) and c), the state of affairs is not so simple. From the mathematical viewpoint, d') is equivalent to a). From the same perspective,

5 The translation into English of a text written in another language may affect some linguistic properties of the original text. Here, the text is simple enough to be translated without substantially changing the features I am taking into account.

b') is not equivalent to a) at all. Nevertheless, it is a *conversational implicature* of a). In the frame of pragmatics, a conversational implicature of a text[6] is the portion of the information provided by the text that follows from the assumption that it is adequate to the context rather than from its propositional content. b') does not follow from the content of a), but from the assumption that a) is appropriate. If b') were false, then a) might be still true, but it would prove inadequate, as a sentence like a') would be much more cooperative. Therefore, the link students recognize between a) and b') does not reside in the propositional content of a), but in the assumption that it is a cooperative contribution to the exchange. It goes without saying that mathematical language[7] is customarily forced to break cooperative criteria, which means that some implicatures cannot be drawn.

Example 2

(2A)	Find a real polynomial p such that:	(a)	the degree of p is 2;
	(b) p has at least one real root;	(c)	p has at least two integer roots.

This example is taken from Ferrari (2001). Problem (2A) is easily solved by almost all Science freshman students, after a short unit on real polynomials. The only source of trouble is the interpretation of 'real' and 'integer', which requires some accuracy, as the adoption of the mathematical use (according to which an integer is a real as well) rather than the everyday-life one (according to which the combined use of the two words may suggest the implicature that 'real' should mean 'non-integer'). If the items (b), (c) are included in a more complex context such as problem (2B), students' behaviors are quite different.

(2B)	Find a real polynomial p such that:	
	(a) the degree of p is 4;	(b) p has at least one real root;
	(c) p has at least two integer roots.	(d) p has at least one complex non-real root.

Problems like be are usually solved by less than 60% of each sample of freshman students. Once more, the main obstacle resides in students' failure in recognizing that any integer root is a real root as well. A good share of students who can apply this property to (2A) seem unable to apply it to (2B). Behaviors like these can be hardly ascribed to the lack of knowledge on integers and reals. More likely, in (2B) students focus on condition (d), who asks for the application of a theorem[8] they regard as important and difficult. This condition is interpreted quite accurately, as most students are not misled by conversational schemes and realize that, though only one non-real root is mentioned, two of them are to be considered. The interpretation of

6 By 'text' I mean any written or spoken instantiation of language of any length, not necessarily a book.

7 Through the paper, by 'mathematical language' I refer to the language customarily used in doing and communicating mathematics at undergraduate level, including verbal and symbolic expressions. In this paper visual representations are not explicitly discussed, although they play a major role in communicating mathematics.

8 "For any real polynomial p, if a complex number z is a root of p, then its conjugate $\bar{z}$ is a root of p too."

‘integer’ and ‘real’ is not taken as a focal point of the problem and is performed according to conversational schemes. Notice that, as often happens in mathematical language, there are only few discourse markers to help the reader to recognize the global organization of the text, including focal points, goals and so on. Most likely, in problem (2A) the lack of an easily recognizable focal point, and the relative shortness of the text, induces students to interpret all of the condition according to mathematical uses. It goes without saying too that these behaviors are common, and usually effective, in everyday-life contexts.

Theoretical implications

The above examples provide us a number of hints I am going to list.

First, they corroborate the claim that troubles heavily involve the verbal component. Second, they point out that the interpretation of a text is hardly a plain translation (based on vocabulary and grammar), but involves the context the text is produced within (including participants and goals). Third, they suggest that the interpretation of texts is a cooperative enterprise which requires the readers (or hearers) to play an active role, performing some inferences, recognizing some part of the text as essential and focusing on them. Fourth, they show that the investigation of single expressions can hardly provide significant insights, but whole texts are to be taken into account; example 2 shows that some expressions may prove more or less troublesome according to the text they occur within. Finally, they suggest that everyday-life and mathematical language are considerably divergent as to use, and that this may prove a severe obstacle to learning.

The last point implies that, as the goal of just preventing students from adopting conversational schemes is of course neither a reasonable nor a viable one, they need to be able to recognize the two ways of using language and to switch between them. This requires some metalinguistic awareness that most often has to be built, as not all students have developed it. All these hints suggest that we need a theoretical frame apt to spot the use-related differences between mathematical language and ordinary one which are relevant to mathematics learning. For these reason we need to borrow ideas and constructions from pragmatics, which fulfils all the above requirements.

A FUNCTIONAL PERSPECTIVE

More precisely, I adopt the frame of functional linguistics, which focuses on functions of language rather than on its forms[9]. The emphasis on functions is quite appropriate because the gap between ordinary language and mathematical one mainly resides in the difference of the functions they play. Mathematical language is not shaped so as to promote interpersonal communication, but rather to provide an effective, well-organized picture of mathematical knowledge and to support the application of algorithms. Anyway, mathematicians, mathematics educators and

9 The main sources in functional linguistics adopted in this paper are Halliday (1985) and Leckie-Tarry (1995).

students must communicate. This may result in using the same words and constructions with different meanings, according to the goal of the text. As the examples above show, the conflict between the interpersonal function of language and the logical, ideational one may hinder students' interpretations processes.

The construction linking texts to contexts is *register*[10]. A register is defined as a linguistic variety based on use. It is a construction linking the situation to both the text, the linguistic and the social system. Each individual can use a register by selecting his or her own linguistic resources. Through the paper, the registers adopted in everyday-life are referred to as 'colloquial', whereas those adopted in academic communication and most books are referred to as 'literate'. Colloquial registers are mostly adopted in spoken communication, although they may be used in writing too, as in informal notes, e-mail or sms messages, whereas literate ones are mostly adopted in written texts, though they may be used in spoken form too, as in academic lectures or some talks between educated people. Literate registers are not necessarily associated with advanced topics nor with high-level linguistic resources nor with the writers or speakers' age. For example, a group of 2nd-graders writing down a report of some complex activity might actually use a literate register.

One of the main claims of this paper is that the registers customarily adopted in advanced mathematics share a number of features with literate registers and may be regarded as extreme forms of them. Some specific features of mathematical registers, such as the violation of cooperation principles, the unfeasibility of most implicatures and the lack of discourse markers have been mentioned above. The example below points out some other aspects.

Example 3

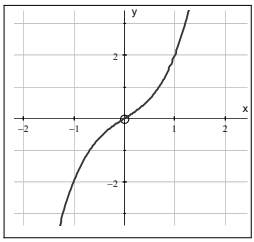
A group of freshman students were required to recognize (and explain) which equation, out of the following

(a) $y = x^3+1$

(b) $y = x^3+x$

(c) $y = x^2+ x$

might match the graph of the function f on the right.



To explain her (right) answer, a student wrote the following text (translated into English verbatim).

> "The graph is increasing and decreasing and passes through 0. I see that x and y >0 and x and y<0. So the graph corresponding to f is the equation (b)."

Texts of this kind are quite common among freshman students. This one is quite inaccurate: the graph is described as 'increasing and decreasing' (which is inconsistent),

[10] Here I adopt Halliday's definition of register, which has been thoroughly discussed by Leckie-Tarry (1995).

it is claimed that it ‘passes through 0’ (in place of (0,0)) and the second occurrence of ‘graph’ is used to mean ‘equation’. The claim that the graph is ‘increasing and decreasing’ might be related to the student’s way of exploring the graph starting from the origin and moving rightwards or leftwards. The expression ‘x and y >0 and x and y<0’ is quite obscure as well. There are two interpretations available. Maybe the first and the third occurrence of ‘and’ are intended to express some logical relationship (such as ‘if x>0 then y>0’). On the other hand, the student when reading ‘x and y >0’ pointed her forefinger to the right side of the diagram, and when reading ‘x and y<0’ pointed to the right; maybe she meant to describe the two sides of the diagram separately, but in writing failed to make this reference explicit through words. In all cases, we are dealing with behaviors common in spoken colloquial registers: relationships between statements are not made explicit through syntax, and a conjunction like ‘and’ is used to express a variety of meanings; references to the context are not made explicit, maybe because in spoken communication the act of pointing or other gestures may get the same goal; words are used quite inaccurately, as often happens in spoken communication, where the addressee can ask for explanation if the meaning is not clear enough; the expressions explicitly defined in mathematical setting (such as ‘increasing function’) are used according to ordinary meaning rather than to the definition; little attention is paid to inconsistency.

All the features of the text suggest that the student in question cannot use literate registers, or, if she can, some reason prevented her from actually using them. As a matter of fact, in literate written registers syntax is a basic way to express meanings, any reference to the context is made explicit through words, words are used accurately, and texts are to be consistent. These features of literate written registers are imposed by a variety of reasons including the need for communicating with people not sharing the same context the text has been produced within, and the need for representing a great amount of complex data with complex relations. In mathematical language, the above-mentioned features of literate registers occur in an extreme form, and, especially if the symbolic notations are involved, there are fewer opportunities of expressing meanings and organizing discourse. The role of syntax, for example, is crucial, as far as often it is the only way to express some meanings. The need for making any reference to the context explicit is even more acute; moreover, there is plenty of words whose meaning has been redefined and that are to be used accurately. On the other hand, despite all the criticism, it is undeniable that the text in question was somewhat effective, as the instructor understood its meaning after all. He had to be much cooperative, and most likely he was expected to be such, as students know that instructors know mathematics quite well. In general, in most teaching contexts, students expect instructors to be cooperative. If communication fails or if the instructor claims that the text is inappropriate, the student might ascribe failure not to his or her product, but to the lack of cooperation by the addressee.

TEACHING IMPLICATIONS

In the previous sections I tried to show that mathematical language shares a number of properties with written literate registers of ordinary language. This means that being familiar with literate registers and their use, which is not a 'natural' condition but has to be built, is a good starting point, if not a prerequisite, to learn to use mathematical language. This raises the problem of the methods more suitable to help students to learn to use literate registers. Of course this cannot be done just at undergraduate level, but a long-term work is needed which should start in primary school. Teaching methods based on grammatical patterns do not work anymore. On the other hand, in standard learning situation students are hardly required to deal with genuine communicative problems. Most often they are required to communicate mathematics to people who already knows it, and whose only task is to evaluate their performance. So we need to design learning situations requiring students to develop suitable linguistic or metalinguistic resources not in conformity to prearranged patterns, but as answers to shared communicative and representational constraints[11].

At college level there are few opportunities to put into practice long term activities aimed at improving linguistic skills. Requiring high degrees of correctness to students with a poor linguistic background just means inducing them to learn by heart or to use stereotyped expressions with no understanding. On the other hand, some linguistic accuracy seems essential in doing and communicating mathematics, and must be developed anyway. To get this, verbal language is to be exploited as a tool to describe and justify procedures, and to gain a better control on performances.

In this frame, discussions between students, at any age level, play a major role, as they provide some of the simplest teaching situations satisfying the conditions stated above. Of course, discussions alone do not produce mathematical knowledge, but nevertheless they may help students to develop linguistic skills that are essential to understand and communicate mathematics, if not to develop mathematical thinking. This requires a shift of emphasis from 'solutions' to verbal explanations and may involve students' and teachers' beliefs and attitudes towards mathematics and mathematics education.

Information technology, if properly exploited, provides a variety of semiotic systems (verbal language, graphs, formulas, tables, …) which allow instructors to design activities requiring interpretation, comparison, conversion and treatment of representations, related to goals explicitly shared by students. Technology provides constraints (e.g., on the format of the data) that are often taken by students more easily than the ones put by the instructors, as they appear as objective requirements rather than decisions subject to the whims and moods of an individual.

11 Ferrari (2002) has shown an example of an activity like that at middle school level.

FURTHER DEVELOPMENTS

The ongoing research on this topic is aimed at refining the comparison between colloquial registers and mathematical ones. This investigation should provide hints on the most appropriate ways of organizing texts intended for students as well as teaching ideas aimed at the improvement of linguistic skills and metalinguistic awareness through the design of teaching methods apt to develop linguistic resources matching the needs of scientific thought without generating needless obstacles. The full exploitation of the opportunities provided by information technology (including the availability of visual representations) is a necessary step to achieve all this. Last but not least there is the goal of making clear the interplay between the use of language and students' beliefs and attitudes towards mathematics and languages.

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ARITHMETIC/ALGEBRAIC PROBLEM-SOLVING AND THE REPRESENTATION OF TWO UNKNOWN QUANTITIES

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We deal with the study of the senses and the meanings generated in the representation of the unknowns in the resolution of word problems involving two unknown quantities. The discussed cases show the difficulties that the students beginning the algebra learning have to deal with when using the equality between "unknown things". For them, applying the equality transitivity property between different (but equivalent) algebraic expressions, or replacing an unknown quantity with its representation in terms of another one is not derived from an extension of the transitivity between numerical equalities or from the numerical substitution. This may have important implications in the algebra problem-solving teaching domain, in which it is usual to take for granted that students spontaneously transfer these numerical issues to the algebraic realm.

Previous research has been undertaken to probe cognitive processes that take place in solving word problems in the transition from arithmetic to algebraic thinking. Bernardz, Radford, Janvier, & Leparge (Bernardz, Radford, Janvier, & Leparge 1992; Bernardz, 2001) have substantially contributed to this research area. Puig and Cerdan (1990) have formulated criteria to determine when a word problem can be considered as algebraic. From a different perspective, A. Bell (1996) has approached this matter by showing through examples how generic problems can provide algebraic experiences that develop manipulative algebraic abilities. Rojano and Sutherland (2001) have studied how technological environment can help students to represent and solve word problems without having to take on board with the algebra symbolic code, from the very beginning.

The present paper addresses the theme of arithmetic/algebraic problem-solving from a different point of view. The results presented are part of the research program "The Acquisition of Algebraic Language " (Filloy and Rojano, 1989; Rojano, 1994), which intends to throw light on the uses of ***Mathematical Sign Systems*** (***MSSs***) (Filloy, 1990) which will culminate in the competent use of the System of Signs of Symbolic Algebra. In previous reports (Filloy, Rojano and Solares; 2003), we have dealt with the problems of the *significations* and *senses* generated in the acquisition of the syntactic abilities needed for the manipulation of what is "unknown" in solving equation processes. In the study we report here , we approach the same theme in the context of the arithmetic/algebraic world problem resolution. We emphasize the role of teaching interventions that promote the use of *Signs Systems* in which suitable strategies for the solution process may be developed.

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THEORETICAL FRAMEWORK

In this study, we adopt the theoretical perspective of ***Mathematical Sign Systems*** (***MSSs***) developed by Filloy, Rojano, Puig (see Filloy, 1990) to analyze the different ways in which pupils specify what is unknown and identify the central relationships of a word problem during the solution process. In previous papers, we have referred to this actions as *logical/semiotic outline* (see for example Filloy y Rubio, 1993). Specifically, we use the theoretical elements that are define below.

The ways to manage what is "unknown" and the resolution methods.- We call ***Method of Successive Analytic Inferences (MSAI)*** the classic analytic method, consisting of the conception of the problem formulation as descriptions of "possible states of the world", and the transformation of such descriptions through logical inferences. What is "unknown" in the problematic situation is interpreted as an *unknown quantity* (a particular number, the age of a person or a measurement of a figure, etc.) The solution is obtained as a "possible state of the world" in which the *unknown quantity* value is described (Filloy and Rubio, 1993).

We call ***Cartesian Method*** (***CM***) the one usually presented in the algebra text books. In our theoretical perspective it consists of the representation of the relationships between the known and unknown elements using a Sign System (that of algebra) "more abstract" than the one in which the problem is formulated, and which allows to transform these relationships into one or several *mathematical texts* the decoding of which gives the solution when returning to the original Sign System.

Mathematical Texts and Teaching Models.- From this theoretical perspective, a ***Teaching Model*** is understood as a problem situation sequence, that is, a ***mathematical text*** sequence T_n (Filloy, 1990), which production and decoding enable to interpret, finally, all the texts T_n is an MSS more abstract, which code makes it possible to decode the texts T_n as messages with a socially well-established mathematical code: the one proposed by the educational aims.

Sense of a method.- During the learning process, the competent use of the method is achieved through the concatenation of the actions triggered by the solution process (Filloy, 1990; Filloy, Rojano and Solares; 2003). Such concatenation, carried out at each stage of the ***Teaching Model***, produces what we call the ***sense of the method***.

THE EXPERIMENTAL DESIGN

We use the methodological framework of the ***Local Theoretical Frameworks*** (Filloy, 1990) in which each specific object of study is analyzed through four interrelated components: (1) a ***Teaching Model***, (2) a ***Cognitive Processes Model,*** (3) a ***Formal Competence Model,*** that simulate competent performance of the ideal user of a ***MSS***, and (4) a ***Communication Model.***

We analyzed protocols from videotaped clinical interviews carried out with 15 13-14 year old pupils from “Centro Escolar Hermanos Revueltas”, in Mexico City. These students had already been taught on how to solve linear equations with one unknown

and related word problems. At the beginning of the study, these pupils showed to have different ability degrees in these algebra topics.

The script of the interview was designed on the basis of a *Teaching Model* that introduces the resolution of word problems involving two unknown quantities through a sequence of *mathematical texts* divided into two blocks:

1. *Abaco Problems* (problems of the form "Find a number that…"), and
2. *Problems in context* (formulated in different contexts, such as geometry and ages calculation)

Based on the ***Formal Model*** analysis the items for blocks (1) and (2) were designed as follows: We designed *Abaco problems* that are associated with two equations systems formed by an equation of one unknown and another involving two unknowns. The *problems in context* designed are associated to systems formed by two equations, both involving two unknowns. Items in both blocks were presented to pupils according to the syntactic complexity (from less to more complex).

RESULTS FROM THE INTERVIEW ANALYSIS.

We analyze interview excerpts from three cases. Two of them, An and Mt are highly competent students in relation to operating one mathematical unknown, and the third one, L, is an average- student in the same terms. These cases allow us to analyze the difficulties that subjects have to cope with problems involving two *unknown quantities*. We shall focus on the *problems in context* block, for in this part of the interview, pupils are required to develop new solution methods as well as new semantics and syntax associated to such methods. On the basis on these three cases, we discuss children's productions in the ***perimeter problem*** and in the ***ages problem***.

Regarding the results found in the *Abaco problems* block, it is important to mention that pupils could identify the two unknown quantities and recognize the two restrictions on their numerical values. In the same block all the subjects deal with the solution of the *Abaco problems* following a route of three steps: (1) finding the unknown value in the one-unknown "equation"[i], (2) replacing the found value in the two-unknown "equation", and (3) finally, finding the value of the second unknown. This route coincides in appearance with the ***Formal Model*** steps to solve this sort of problems. Nevertheless, what children do is to follow the route using a variety of MSS strata, such as natural language, arithmetic or algebra, and not necessarily that of the manipulative algebra. Due to the regularity with which this spontaneous strategy manifests itself and to the importance it attaches later, we will call it strategy ***S(1,2)***.

The perimeter problem:

> *The perimeter of a rectangle is 5 times its width, the length measures 12 meters ¿How much does the width measure?*

This is the first problem in the interview in which the two equations include both unknowns. The subjects translate with no difficulties the problem into the algebraic

language and find and equation system of the form: ***P = 5a*** and ***P = 24 + 5a*** They interpret these expressions as texts about ***unknown quantities***, the measurement of the width "***a***" and the measurement of the perimeter "***P***". After solving the *Abaco problems block*, they realize that if they find the value of one of theses ***unknown quantities***, they will be able to find the value of the other one and solve the problem.

The strategy S(1,2) and the two symbols.- L intends to apply strategy S(1,2), but after a number of syntactic transformations performed on the equations she notices that it is not possible to apply such strategy:

L: I want to, I mean I want to solve the... I want to solve this equation. **L:** This is going to change from this side to the other side, subtracting. But I have two symbols: "***P***" and "***a***". **L:** Look, I want to find the value of "***a***", but... ¡Ups! No, I don't know….	She points out the equation she previously wrote: ***24 + 2a = P***. She points out ***2a***. She points out ***2a*** and then writes: ***24 = -2a***

The appearance of the two ***unknown quantities*** in both equations does not allow her to apply strategy ***S(1,2)***, which previously allow her to solve all the previous problems. This phenomenon appears in all the cases analyzed. Two main algebra concepts that children will build up later on will allow them to apply this strategy. The concepts in question are: ***algebraic transitivity*** and ***equality between algebraic expressions***.

Algebraic transitivity and numerical transitivity.- Unlike what is taken for granted in algebra traditional texts books, the ***transitivity between algebraic expressions*** is not derived from a simple extension or generalization of the ***numerical transitivity***. For the cases we analyze, applying the transitivity property to the equations of this problem makes sense only when such property is referred to the problem context. This issue is illustrated with the following excerpt.

Mt wants to find the value of the width (represented by "***x***" in her case). She has written these set of equations: ***P = 5x*** and ***P = 24 + 5x***.

I: ¿Don't you figure out how could we get the value of "x". **Mt:** Well, yes. Let's see ... "8" plus "24"... makes "16"…No! Okay, I have already found out how! **Mt:** So, now we have two "***x***" plus 24. **Mt:** Here we have "***x***" two times and here we have "***x***" five times.	 Mt does some calculations verbally. Mt writes down: ***8***. This is the value of the width that solves the problem. Mt writes the equation: ***2x + 24 = 5x.*** Here Mt equalize the two expressions she has for the perimeter. Mt points out ***2x*** and ***5x***.

For Mt establish apply the ***transitivity*** to the expressions for the perimeter only has sense if there is a number that makes the equation true. In fact, Mt refuses to write the equation among the expressions and only does it once the width numerical value is found.

In short, to solve this problem, pupils carry out a series of *analytic inferences* (MSAI) that give sense to the algebraic expressions equalization. For them, the ***unknown***

quantities have their referents, whether in the rectangle drawing corresponding to the problem or in the number that makes true the corresponding numerical equation. This tendency to be attached to the problem context makes in these cases that the ***transitivity*** is carried out for quantities (not for algebraic expressions).

The ages problem:

> *The difference between the ages of Juan and Carlos is of 12 years. In 4 years time Juan, who is the oldest, will double Carlos' age. Which are the current ages of both?*

This problem solution represented new difficulties for the interviewees. Its syntactic complexity demands not only high translation skills, but also to apply the *sense* of the actions recently generated: the use of one unknown expression given in terms of another unknown factor.

Manipulating the unknown through the MSAI.- An resorts to his high logical analysis competencies to solve this problem. Although he is highly competent in manipulating one unknown, he does not have yet the necessary syntactical competencies to algebraically manipulate the two unknowns of the problem. An uses the algebraic code to represent the relationship between data and unknowns, and understand them as numerical relationships between *unknown quantities* (Juan and Carlos' ages). These *mathematical texts* allow him to analyze the relationships by applying MSAI. He spontaneously carries out the following reasoning:

An: The difference between Juan and Carlos' ages is of 12 years. Juan, I'll name him "***J***", then you have that "***J – C = 12***" and we also have that "***J + 4***" will be equal... yes!, "***J + 4 = (C + 4) 2***".	Here he finds the associated system of equations: ***J – C = 12*** and ***J + 4 = (C + 4) 2***.	***Observations.*** *An's reasoning for finding the current ages is the following: After translating the problem into the algebraic language, he realizes (as an obvious implication of the problem context) that the difference between Juan and Carlos' ages will be always of 12 years. Specifically, in 4 years time it will be of 12 years. But he also knows that in 4 years time, Juan's age will be the double of Carlos' age (up to that date). So, he have that in 4 years time Juan's age will be: (1) Carlos' age up to that date plus 12 years (because the ages difference is always 12 years), but it will also be (2) two times the age of Carlos(up to that date). Then, he deduces that one time the age of Carlos(up to that date) has be equal with 12 years.*
An: Then, lets first see which we can get..., we get..., I've got it, wait, Juan's age..., no here.	He points out the equation: ***J + 4 = (C + 4) 2***.	
An: I have to get that 12 will be the difference, even, of these two.	He points out ***J + 4*** and ***C + 4*** in the equation: ***J + 4 = (C + 4) 2***.	
An: So Carlos' age up to that date would be 12…	He points out ***J + 4*** and ***C + 4***.	
An:…and the age of Juan would be 16 in 4 years time. No!, it wouldn't be 16. The age of Juan would be 24 in 4 years time.		
An: So, from it I get that Juan is now 20 years-old, and Carlos is 8 years-old, the difference is 12… and this other thing I have would be 20+4 equal to (8+4)2, 8+4 is equal to 12, 2 times 12 is equal.	He refers to the equation: ***J – C = 12***.	*Then, up to that date, Juan's age will be 24.From that, he obtains that the current ages are 20 and 8.*
	He points out the equation: ***J + 4 = (C + 4) 2***. He makes the calculations verbally.	

An carries out the series of *analytic inferences* from the Natural Language Signs System. He does not need to use any strictly algebraic transformation nor signification. His solution requires a complicated logical analysis and an intense use of the working memory.

Actions sense and the need for the mathematical unknown notion[ii]***.-*** L and Mt did not show to have the same logical analysis level as An in the ages problem. They require the interviewer's intervention to overcome the difficulties encountered in this problem. Introduction of the ***mathematical unknown*** notion was necessary. Lets see how it is done in L's case.

She has not acquired the competencies to interpret Carlos and Juan's ages as ***mathematical unknowns***, which would allow to syntactically manipulate them. At the outset, she interprets them as ***unknown quantities***, which she cannot operate because she do not known its value.

With the intervention of I, L has translated the problem to the system of equations: $J - C = 12$ and $J + 4 = 2\ (C + 4)$.	
I: Here you have that Juan's age minus Carlos' age equals 12. If we had here a number, for example 10, could you find Juan's ages value?	I points out the equation: $J - C = 12$ I writes 10 below C in that equation.
L: Yes! Adding these two.	L points out 10 and 12.
I: Ok. So, if we knew Carlos' age, could we calculate Juan's age?	
L: Yes!	
I: How?	
L: Adding Carlos' age to 12.	
I: Then, if I represent Carlos' age with C, how much is the value of Juan's age?	I points out C and J in the equation: $J - C = 12$.
L: <u>I don't know it yet.</u>	

At this moment, the teaching intervention was necessary to favor the strategies and interpretations that allow to solve the problem:

I: How much would "J" value? It is a number, a number you don't know, but it is a number. Suppose that I already know what number is it, but I do not want to tell you. I'll call it "C". I tell you: it is "C". Then, how much is the value of Juan's age?	
L: "C" plus 12.	L writes the equation: $J = C + 12$

Once the ***mathematical unknown*** notion was introduced, L solves the problem picking up the ***sense*** learned in the previous problem (the ***perimeter problem***):

L already got the equations: $J = C + 12$ and $J + 4 = 2\ C + 8$. Below them she writes the equation: $(C + 12) + 4 = 2C + 8$.	
I: What did you do? Why did you put that here?	I points out $C + 12$ in the equation: $(C + 12) + 4 = 2C + 8$.
L: So that... Do you remember when "P"... when here... we replaced it with "$5a$"? Well, here <u>I substituted "J" by "C plus 12", so that there are only "C's"</u>.	L points out to the perimeter problem. She points out the J in the equation: $J + 4 = 2C + 8$, and $C + 12$ in the equation: $(C + 12) + 4 = 2C + 8$.

L makes a substitution in accordance with the (intermediate) sense learnt in the ***perimeter problem***: "*I substituted "J" by "C" plus 12 so that there are only "C's"*". L's actions are addressed by the application of the strategy S(1,2).

Although, in the cases we analyzed, introducing the notion of ***mathematical unknown*** was crucial, this notion had to be interpreted by the subjects in terms of concrete referents (those of the arithmetic MSS), in order to ***manipulate an unknown factor in terms of the other*** or to apply the ***transitivity*** property to "algebraic expressions". In fact, in this new transition step (when students for the first time have to deal with problems and systems of equations involving two unknowns), building up more abstract notions involves interpretations of such notion at a more concrete level.

FINAL REMARKS AND TEACHING IMPLICATIONS

Issues arising from this study suggest that dealing with problems and equations involving two unknowns quantities imply the necessity to re-elaborate: (a) the notion of *mathematical unknown*, (b) the notion of *algebraic equality*, and (c) the notion of *algebraic representation of the unknown*. Nevertheless, traditional teaching models introduce the problems with several unknown quantities, assuming that, once the first elements of the algebraic language are acquired, the further development of algebra will be carried out because of a simple extension of what was previously learned: *one unknown representation and manipulation*.

Specifically, some of these results show that applying the equality transitivity property between different (but equivalent) algebraic expressions, or representing an unknown in terms of another one , are not derived from a generalization of the numeric equality transitivity property, nor from the numerical substitution.

Additionally, advantages and difficulties of the MC are shown. The application of the MC allows to translate the sentences given in the Natural Language Signs System into *mathematical texts* given in the algebraic MSS and then to apply syntactic transformations in said SS . In this way, it is prevented that the significations specific of the problem context interfere the final analysis stage of the resolution process and the operation of the unknown is enabled.

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[i] With ***"equation"*** we refer to a relationship between known and unknown quantities expressed in natural language and not to the algebraic equality called ***equation***, as it is the case of the Abaco's problems which are entirely rhetoric.

[ii] Here we make the difference between the terms ***unknown quantity*** and ***mathematical unknown***. With ***mathematical unknown*** we refer to a value which, once substituted in an equation or set of equations, makes true the restricted equalities. That is, the notion ***mathematical unknown*** belongs to the realm of symbolic algebra.

EQUITY AND COMPUTERS FOR MATHEMATICS LEARNING: ACCESS AND ATTITUDES

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Equity and computer use for secondary mathematics learning was the focus of a three year study. In 2003, a survey was administered to a large sample of grade 7-10 students. Some of the survey items were aimed at determining home access to and ownership of computers, and students' attitudes to mathematics, computers, and computer use for mathematics learning. Responses to these items were examined by several equity factors (gender, language background, socio-economic status, geographic location, and Aboriginality), by grade level, and by mathematics achievement self-ratings. Equity factors were more salient with respect to computer ownership than with attitudes. Attitudes to computers for mathematics learning were more strongly related to attitudes to computers than to attitudes to mathematics.

INTRODUCTION AND PREVIOUS RESEARCH

The use of technology, including computers, is widely believed to be beneficial to students' learning of mathematics (Forgasz, 2003). The Victorian (Australia) government has instigated the *Bridging the digital divide initiative* (Department of Education and Training [DE&T], 2002a) which is said to ensure:

> equity of access to information and communication technology for all students, regardless of socio-economic or geographic disadvantage. The 2001/02 State Budget provided $23 million over three years for additional computers and networking… to bring all schools to a 1:5 computer to student ratio… (DE&T, 2002a, p.1)

Hence, it is expected that for students in Victorian government schools today there should be access to computers for learning across the curriculum, including mathematics. Through the *access@schools* program, schools in regional and rural Victoria (that is, non-metropolitan areas) are said to have been enabled "to provide their local communities with free or affordable access to the Internet and to their information and communication technology (ICT) facilities" (DE&T, 2002b, p.1).

Government initiatives such as those described above are to be applauded. Yet, findings from previous research would suggest that equity issues with respect to education generally (e.g., Teese, Davies, Charlton, & Polesel, 1995), and to mathematics teaching and learning in particular (e.g., Allexsaht-Snider & Hart, 2001), are complex. That is, by simply providing more computing equipment and cheaper access to ICT, it cannot be assumed that equity of access will automatically result, a view expressed in the UK by Selwyn, Gorard and Williams (2001).

Previous research findings on attitudes to computer use in education have also revealed inequities. Forgasz (2002) summarised a number of gender differences favouring males including: enjoyment, perceived competence, views on usefulness, parental encouragement, personal computer ownership, tertiary course enrolments,

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programming, and game playing. Based on three sets of items included in a survey questionnaire administered to a large sample of grade 7-10 students, Forgasz (2002) found that the students did not appear to stereotype mathematics as a *male domain*, held beliefs about computers that were consistent with the traditional perception of male technological competence and female incompetence, but were a little more ambivalent when computers were associated with the learning of mathematics. Students were reported as being less convinced than their teachers that computers help mathematical understanding; female students were less convinced than their male peers, and no differences in beliefs were noted by student ethnicity or socio-economic status (Forgasz, 2003). According to Hanson (1997), however, computer use is not an educational panacea but exacerbates inequities with respect to race/ethnicity and socio-economic status [SES]. In contrast, Owens and Waxman (1998) reported greater computer use by African American students than by white and Hispanic students and postulated that positive attitudes explain the findings.

Using an attitudes instrument that they developed, Galbraith, Haines and Pemberton (1999) found that their computer-mathematics subscale correlated more strongly with computer confidence and computer motivation than with the equivalent mathematics measures; no equity dimensions were considered in this research study.

The extent to which students in Victorian schools have access to computers for mathematics learning at school and at home is not known. Whether equity of access, based on gender, socio-economic status, ethnicity, and geographic location, has been achieved is also not known. The relationships between students' attitudes to mathematics, to computers, and to computer use for mathematics learning have not been examined by this same range of equity factors. For this paper, data on these issues have been explored and the results presented and discussed.

AIMS AND METHODS

The focus of the three-year study from which findings are reported in this paper was on the use of computers for the learning of secondary level (grades 7-10) mathematics. In summary, the research design for the three years included:

Year 1: surveys of mathematics students in grades 7-10 and their teachers; survey of grade 11 students reflecting on previous use of computers for mathematics learning – 29 schools were involved.

Year 2: in-depth studies of grade 10 mathematics classrooms at three schools – surveys, observations, interviews.

Year 3: repeat of Year 1 surveys in same schools – only 24 schools participated.

If students are to benefit from using computers for mathematics learning, they need to be able to access them, as required, both in school and at home. One of the aims of the present study was to establish the extent to which students do have such access and whether there are any issues of equity with respect to that access.

Included in the survey questionnaires administered in 2001 and in 2003 were questions about: computer organization in schools and use of them for mathematics learning; and computer access at home. Data were also gathered from students on a range of equity factors including: gender; socio-economic status (SES[1]); two dimensions of ethnicity – language spoken at home (ie. ESB/NESB[2]), and Aboriginality (ATSI[3]); and geographic location of school attended (metropolitan or rural). Grade levels and students' self-ratings of mathematics achievement level (SMA) were other variables considered important for analysis. To determine self-ratings of mathematics achievement, students were asked to rate their mathematics achievement levels on a 5-point scale: 5=excellent to 1=weak.

Previous research findings have revealed differences in attitudes towards mathematics on a range of equity factors. Thus, another aim of the present study was to measure students' attitudes to mathematics (AM), to computers (AC), and to using computers for learning mathematics (ACM), and to examine if there were differences in these attitudes on the same range of equity factors as for computer access. It was also considered important to identify the relationships between the three attitude measures to determine if students' attitudes to the use of computers for mathematics learning were more strongly related to attitudes to computers or to attitudes to mathematics. The survey questionnaire included three clusters of eight Likert-type items with 5-point response formats (strongly disagree to strongly agree) to measure these attitudes. The eight AM items were drawn from previous instruments. Slight wording modifications to some of the AM items, and items drawn from other instruments (e.g., Galbraith et al., 1999) made up the other two clusters of items.

For each cluster of items, a reliability check and a principal components factor analysis were conducted to determine if the items formed a uni-dimensional scale. As a result of the analyses, poor items were eliminated. The characteristics of the three resulting attitude scales, with sample items, are summarised in Table 1.

Attitude scale	Items	Sample items	Alpha	Mean[a]	SD
Mathematics (AM)	7	I enjoy mathematics	.745	3.70	.70
Computers (AC)	5	I feel confident using computers	.722	3.35	.76
Computers for mathematics (ACM)	5	Using computers helps me learn mathematics better	.756	3.15	.74

[a] For ease of comparison, the mean shown is the scale mean divided by the number of items in the scale.

Table 1: Summary characteristics for the three attitude scales

[1] SES was determined from postcodes (zip codes) found in ABS (1990).

[2] NESB (Non-English speaking background): defined by positive responses to: "Do you regularly speak a language other than English at home?"

[3] ATSI = Aboriginal or Torres Strait Islander

RESULTS AND DISCUSSION

The sample

In 2003, the sample comprised 1613 grade 7-10 students from 24 schools in the state of Victoria, Australia. There were approximately equal numbers from each grade level: 425 (26%) Gr.7, 415 (26%) Gr.8, 396 (25%) Gr.9, 377 (23%) Gr.10. More than half of the students, 917 (57%), attended schools in metropolitan Melbourne.

In Table 2, the composition of the sample is shown by a range of equity factors – gender, socio-economic status (SES), language background (NESB), and Aboriginality (ATSI); response frequencies and valid percentages are shown.

Gender		SES			NESB	ATSI
F	M	High	Medium	Low		
810 51%	794 49%	251 16%	914 59%	390 25%	359 22%	28 2%
N: 1604		N: 1555			N: 1607	N: 577

Table 2: Grade 7–10 students by equity factors

The data in Table 2 reveal that there were approximately equal numbers of females and males, most students (≈60%) were from medium socio-economic backgrounds, about a fifth (22%) of the students speak a language other than English at home, and that a very small minority (2%) was Aboriginal. The sample profile is not inconsistent with 2001 Australian census data in which it was found that: 40% of Australians live outside capital cities; 0.5% of Victorians identified themselves as ATSI; and 75.3% of Victorians reported being English speakers at home (Australian Bureau of Statistics [ABS], nd). Based on ABS (1990), the proportions of high, medium, and low SES backgrounds in the state of Victoria are: 19%, 59% and 22% respectively. Thus, the SES profile of the sample was also representative of the population of the state of Victoria.

Self-ratings of mathematics achievement (SMA)

The mean self-rating of mathematics achievement was 3.61. There was a statistically significant difference by gender: F= 3.49, M=3.74; t=-5.86, p<.001. The frequencies (and percentages) of the achievement self-rating levels are shown in Table 3. It can be seen that the vast majority of students considered themselves average or better in mathematics.

	5=Excellent	4=Good	3=Average	2=Below average	1=Weak
N=1608	228 (14%)	689 (43%)	571 (36%)	81 (5%)	39 (2%)

Table 3: Students' self-ratings of mathematics achievement

Access to computers at school and home, and student ownership of computers
According to the teachers of the students who also completed a survey instrument, each school had computing resources, regardless of its geographic location or its socio-economic categorization. Computer laboratories were found in all schools and several also had a single computer or clusters of computers in classrooms. About 52% of the students reported having a CD-ROM accompanying their mathematics textbooks; 46% of their teachers said that students had used the CD-ROMs. The survey was administered half way through the academic year. At that time 63% of the students reported having used computers in mathematics classes that year and 79% reported having used computers for mathematics in earlier years of schooling.

Of the 1533 students responding to the item computer access at home, 97% (1487) indicated that there was at least one computer available to them; 53% (808) reported having at least two computers. For those with at least one computer at home, the extent of student personal computer ownership (frequency and related percentage) by equity factors is summarised in Table 4. Chi-square tests were conducted to test for statistical significance by each equity factor. The results are also shown in Table 4.

Gender		SES			Language		Aboriginality		Location	
F	M	Hi	Med	Lo	ESB	NESB	ATSI	non-ATSI	Metro	Rural
249 33%	341 47%	184 50%	294 35%	88 38%	410 36%	177 55%	13 57%	559 39%	413 49%	179 28%
p<.001		p<.001			p<.001		ns		p<.001	

Table 4: Student computer ownership by equity factors and χ^2 results

As shown in Table 4, for all equity factors other than Aboriginality, there were statistically significant differences in the proportions of students owning their own computer. A higher proportion of males than females, of high SES than medium and low SES students, of NESB than ESB students, and of students attending metropolitan than rural schools owned their own computers.

It was not surprising to find that more high than medium or low SES students owned computers. Since more wealth is found in Australia's large cities than in rural areas, it was also not unexpected to find that students at schools outside metropolitan areas were less likely to own computers. The known migrant phenomenon of 'aspiring to upward mobility' may explain the higher computer ownership rates among NESB than ESB students. That parents are more likely to purchase computers for their sons than their daughters supports previous research results. This finding is of concern as it reflects a pattern of stereotyping that Australian educators no longer expect to find.

The results of the chi-square tests also indicated statistically significant differences in the proportions of students owning computers by grade level ($p<.001$) and by self-ratings of mathematics achievement ($p<.05$). Computer ownership increased with grade level (Gr.7: 32%, Gr.8: 37%, Gr.9: 45%, and Gr.10: 46%) and was highest

among those who considered themselves *weak* at mathematics, followed by those who considered themselves *excellent*: *weak* - 58%, *excellent* - 47%, *below average* - 43%, *good* - 39%, and *average* - 37%. The second of these findings was unexpected. Perhaps some parents believe so strongly that computers will help their offspring educationally that they are prepared to buy computers for children whose mathematics achievement levels are expected to improve as a result of the purchase.

Attitudes to mathematics, computers, and computers for learning mathematics
Mean scores on the three attitude scales by the various equity factors, by self-ratings of mathematics achievement, and by year level were compared using independent groups t-tests or one-way ANOVAs as appropriate. The results are summarised in Table 5 - space constraints precluded inclusion of mean scores in each sub-category.

	Gender	SES	Language	Aboriginality	Location	SMA	Grade
AM	M>[a]***	Hi>***	NESB>***	non-ATSI>***	Metro>***	5>***	7>*
AC	M>***	ns	NESB>*	ns	ns	5>***	7>***
ACM	M>***	ns	NESB>*	ns	ns	5>***	7>***

[a] M> means that Males scored higher on average than did Females

* p<.05 ** p<.01 ***p<.001 ns=not significant

Table 5: Attitudes by equity factors: t-test/ANOVA results

The data in Table 5 indicate that for all three attitude measures, males, NESB students, students who consider themselves *excellent* at mathematics, and grade 7 students consistently held more positive attitudes than their peers in the respective equity categories. There were also statistically significant differences in mean scores by each equity factor on the AM scale and fewer statistically significant differences were found for the other two, similarly behaving, attitude scales.

Relationships among the attitude scales
Pearson bi-variate correlations between the three attitude scale measures, students' self-ratings of mathematics achievement (SMA), grade level, and student SES are shown in Table 6.

	AC	ACM	SMA	Grade level	SES
AM	.20*	.18*	.58*	-.07	.06
AC		.57*	.15*	-.11	.02
ACM			.12*	-.25*	.04
SMA				-.03	.06
Grade level					-.06

* p<.01

Table 6: Bivariate Pearson product-moment correlations

The bivariate Pearson product-moment correlations found in Table 6 reveal: moderately high correlations between AM and SMA (0.58) and between AC and ACM (0.57); small positive correlations between AM and AC, AM and ACM, and between AC and SMA; a small negative correlation between ACM and grade level; and no significant correlations at the $p<.01$ level with SES.

The high correlation between AM and SMA (0.58) supports earlier reported findings relating mathematics achievement and positive attitudes, particularly confidence, towards mathematics (e.g., Leder, 1992). The high correlation between AC and ACM (0.57) and low correlation between AM and ACM (.18) mean that attitudes to computers for mathematics learning are more closely associated with attitudes to computers than to attitudes to mathematics. These results are consistent with Galbraith et al.'s (1999) findings with tertiary mathematics students. These correlations need to be monitored and explanations found for them.

The small negative correlation between ACM and grade level supports previous findings that younger students are more positive about mathematics than older students (e.g., Cao, Forgasz, & Bishop, 2003). The finding of no significant correlations with student SES is important in that it suggests that attitudes towards mathematics, computers, and computers for mathematics learning, do not seem to be affected by the inequity of SES in student personal computer ownership.

FINAL WORDS

In summary, the findings reported in this paper indicate that there are equity issues associated with grade 7-10 students' personal computer ownership and with their attitudes towards mathematics, computers, and computers for mathematics learning. As discussed above, many of the findings reported here were consistent with earlier published research results.

Interestingly, SES and geographic location were equity factors implicated in computer ownership and in attitudes to mathematics, but not in the two attitude measures associated with computers. Compared to their respective counterparts, males, students from non-English speaking backgrounds, and students with higher self-ratings of mathematics achievement, appear advantaged with respect to computer ownership as well as holding more positive attitudes on all three attitude measures. Based on previous research linking attitudes to participation (e.g., Leder, 1992), they are the ones more likely to persist with higher level studies in mathematics.

That the attitudes to computers for mathematics scale was found to be more highly correlated with the attitudes to computers scale than to the attitudes to mathematics scale raises a number of issues worthy of further research. As computer use becomes more widespread in mathematics classrooms, what will be the impact on students' attitudes towards mathematics and to their longer term participation in mathematical studies? Will there be differential effects that re-inforce or challenge more traditional patterns of disadvantage with respect to mathematics learning outcomes? What will become of those who hold less positive attitudes towards computers? Issues of equity

and affect associated with computer use in mathematics classrooms must not be ignored if students' opportunities to learn mathematics are to be optimised.

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THE TACIT-EXPLICIT DYNAMIC IN LEARNING PROCESSES

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In this report we present the methodology used in a study that investigated the tacit-explicit dynamic in learning processes. We have analyzed an episode related to a discussion about the difference between plane figures and spatial figures promoted by the teacher in her mathematics classroom (the students are aged 11 to 12). The data analysis was based on some aspects of Polanyi's theory on tacit knowledge, and benefited from a variation of the 'graph-theoretical model for the structure of an argument' developed by Strom, Kemeny, Lehrer and Forman. The methodology employed exhibited a strong indication that the lack of correspondence between what the students are uttering and their original understandings is related to the tacit-explicit dimension.

INTRODUCTION

In a recent study Frade and Borges (2002) analyzed some current curricular goals in the light of Ernest's (1998b) model of mathematical knowledge, according to its mainly explicit and mainly tacit components. The materials examined were suggested by curricular documentations from several countries and at different levels of teaching. The analysis showed the prevalence of the mainly tacit components over the mainly explicit in such curricular goals.

Since then, we have been working on this subject aiming at a better understanding of the tacit dimension of mathematics teaching and learning. Throughout our process of investigation we found different references to Polanyi's (1962, 1969, 1983) concept of tacit knowledge. Fischbein (1989), Tirosh (1994) and Sternberg's (1995) researches address what can be called 'Polanyi's psychological version of tacit knowledge': knowledge that functions as subsidiary to the acquisition of other knowledge. On the other hand, Ernest (1998a,b) and Wenger (1998), for example, use the words 'tacit' and 'explicit' as opposites to refer to different, but complementary dimensions of the same component of a certain practice. Whatever the case, the above-mentioned authors share in some way Polanyi's epistemological thesis that all knowledge is tacit or constructed from tacit knowledge (Polanyi, 1969).

In this report we present the methodology employed in the analysis of an episode related to a discussion about the difference between plane figures and spatial figures, promoted by the teacher in her mathematics classroom (the students are aged 11 to 12). Such analysis benefited from a variation of the 'graph-theoretical model for the structure of an argument' developed by Strom, Kemeny, Lehrer and Forman (2001) to integrate those two above-mentioning meanings of the concept of tacit knowledge with other important element of Polanyi's theory: the three areas or domains – ineffable, intermediary and sophistication – in which the relation between thought and speech varies from one extreme: tacit prevailing over the explicit, to another:

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tacit and explicit falling apart, moving through an intermediary level: tacit corresponding to the meaning of speech (Polanyi, 1962, p.87). This integration allowed us not only to investigate the types of knowledge – mainly explicit or mainly tacit – the students used in a 'psychological way' to perform the task of elaborating an understanding of the difference between plane and spatial figures, but mainly how much the projection of those types of knowledge on the task were manifest tacitly or formalized by the students. Some results of the research are presented.

DEVELOPMENT AND METHODOLOGY

Initially, the students were asked to elaborate and present their understanding in writing (this was considered as task 1). To this end, they had to observe a classifying table – flat, plane, volumeless forms *versus* spatial forms that can have a volume – proposed in their textbook. This table had solely pictures. After some time, the teacher conducted a conversation about the students' understanding of the difference between plane and spatial figures. When the conversation began, some students manifested difficulty in putting their understandings in writing. So, the teacher let them elaborate such understandings orally in real time (this was considered as task 2).

The episode lasted for twenty minutes and was recorded on tapes, which were transcribed entirely. After examining the data categories were established to account for: 1) students' knowledge used in a subsidiary way to perform and control of the task, 2) students' internal articulation that preceded their utterances, 3) the teacher's interventions, 4) two other non-fully observable processes: concentration (observation of the classifying table by students) and shifting of focus (not recognition of some knowledge as instruments by the students). Below I illustrate the categories.

1. Students' tacit knowledge (related to the task in question)

C1: Surfaces

Example: "…in some figures **there are some flat forms that make** a figure with a volume. Example, the **cylinder** has two **faces** with the form of a **circle**, the **prism** has two **faces** of one **hexagon** and two faces of a **rectangle**." (task 1)

C2: Capacity

Example: "The difference between the flat, the plan and without volume and the spatial figures that can have a volume is that the flat ones **cannot hold material inside** and the ones which have volume **can hold material inside**. " (task 1)

C3: Width

Example: "This one is **hollow inside** and this one **is not hollow inside**". (task 1)

C4: Rigid movement

Example: "…if you take one of these triangles and **take it from the paper page** it will be just like the page; you **turn it** and it is all the same. Now, this one will be like a

pencil, you **turn it** and **it shows other angles of vision**. I think that's the main difference." (task 2)

C5: Bending

Example: "We put here that the figures without volume **do not stand** and the spatial figures **do**." (task 1)

C6: Tangible reality

Example: "This ones (referring to spatial figures) **are real things** and **those ones are papers** (referring to plan figures)". (task 2)

C7: Meta-cognition

Example: "Ah, I **more or less understood** what some of them are". (task 2)

The words or expressions in bold in those utterances indicate our identification of the clues the students gave about the knowledge they were using in a subsidiary way to elaborate an understanding of the difference between plane and spatial figures. The categories were named according to this knowledge. This does not mean that the students were conscious of having that knowledge or even that some of those types of knowledge stood as formalized mathematical knowledge. Also we do not state affirmatively what the origin of that knowledge was: school instruction, informal acquisition through daily life experience, or germinal mathematical ideas. We have interpreted the above-mentioned knowledge C1, C2,…,C6 as mathematical because they were used tacitly or instrumentally for a mathematics task. Besides, it is possible that in the future that knowledge can be formalized mathematically allowing the students to take them as mathematical knowledge.

Task 1 in parenthesis indicates that the student was reading what he/she wrote. Task 2 in parenthesis indicates that the student was elaborating his/her understanding in real time.

2. Students' internal articulation of the understandings produced

E1: Priority of tacit - identified with Polanyi's ineffable domain of co-operation between tacit (personal) and explicit (formal). E1 indicates an internal articulation, which was not projected in speech or else was projected vaguely and not precisely. Examples of utterances that we have interpreted as resulting from E1:

"It's the word that doesn't come out". (task 1)

"It's like a piece of paper, you turn it that's all it has". (task 2)

E2: Tacit on the borderline with the explicit. E2 indicates an internal articulation which was progressively projected in speech in such a way that the tacit seemed to be close to the explicit. Example:

"Look here, all the spatial forms that have volume give an example of being real. And to be real (some hesitation) look at a prism, for example, of a hexagon. You connect a hexagon to another with rectangles you have a prism". (task 2)

E3: Tacit coincides with explicit - identified with Polanyi's intermediary domain of co-operation between tacit and explicit. E3 indicates an internal articulation, which was fully, and exactly projected in speech. Example:

> "We put here that the figures without volume do not stand and the spatial figures do". (task 1)

E4: Explicit separate from tacit - identified with Polanyi's sophistication domain of co-operation between tacit and explicit. E4 indicates an internal articulation which was not reflected in speech. In this case, although the speech was confident and with no hesitation, it was incoherent or contradictory. Example:

> "The forms without volume can only be seen in one way, they are plane and flat. The forms with volume can be seen in many ways, almost all of them are solid and have a volume". (task 1)

E5: Explicit under check. E5 indicates an internal articulation, which results from the students mobilizing his/her meta-cognition. The student performed the task but doubted the relation between his/her internal articulation and its external representation. Examples:

> "I wrote here, but I don't know if it is correct". (task 1)

> "What is this?! The hexagon has the surface that has the base for, the base?!" (task 1)

Those categories were built in the following way: from the original records and their transcriptions we searched for students' utterances excerpts, which could be interpreted as a result of their *modus operandi*. On the other hand, each internal articulation identified would be preceded by the mobilization of specific tacit knowledge, which made up students' understandings.

3. Teacher's interventions

I1: Commands

> Example: "Now, you all must observe the classifying table for some time and then write the difference between …"

I2: Guiding the speeches

> Example: "And you Peter, what about the difference between…?"

I3: Explaining

> Example: "This doesn't have to be right, it is your perception."

I4: Demanding explanation

> Example: "What do you mean by…?"

I5: Posing question

> Example: "There is more to it. Can't you see anything else?"

I6: Listening

> Example: "Go on"

The teacher's mental actions were not examined in this research to avoid more risk of interpretation and because the focus of the analysis was the students' learning processes.

4. Two other non-fully observable processes

O: Concentration, which corresponds to the students' observation of the classifying table, and SF: Shifting of focus, which corresponds to an interruption of the students' performance due to an unfamiliar feeling with the instruments that they or their classmates had used to complete the task.

THE GRAPHIC REPRESENTATION OF THE DATA

Once those groups of categories were constructed, we divided the episode into five segments where each of them was represented by a graph (e.g, graph 1 corresponds to segment 1, and so on). The idea of the graphs came from the work by Strom et al (2001), and was intended to exhibit the tacit-explicit dynamic of the episode.

In each graph (refer to table 1 as an example of all graphs) the categories were disposed in a circle and oriented flows were drawn to represent the dynamic among such categories (or the structure of the actions produced during the event). From each graph the episode was, then, re-analyzed.

Continuous flows (straight lines) represent the observable aspects of the episode: the students' and the teacher's utterances. Interrupted flows (dotted lines) represent non-observable aspects. For example, in the graph 1, the continuous flow number 11, which departs from I2 toward C7 corresponds to an utterance of the teacher (observable aspect) to a student, and resulting from an intervention of the type I2: Guiding the speech. This teacher's utterance, in its turn, provoked in that student or in another student the mobilization of his/her tacit knowledge of the type C7. The next flow – number 12 – in dotted line, indicates that the student attributed meaning to and integrated such knowledge (non-observable aspect) producing an internal articulation about his/her understanding of the difference between plane and spatial figures, which was not projected in his/her utterance or was projected vaguely and non-precisely (in the former, although the student had produced an utterance it did not project any clue about his/her understanding of the difference between plane and spatial figures). This utterance (observable aspect) was then originated from E1 and was represented by the flow number 13 in straight line.

The darker flows correspond to the students' actions in relation to task 1. For example, the flow number 34, which departs from E3 toward MF, corresponds to a student's utterance resulting from his/her articulation of the type E3: an internal articulation that was fully and exactly projected in his utterance, and answering the task 1. The orientation of that flow shows that this utterance provoked in him/her or in another student a shifting of focus. Following, the flow number 35 indicates that such a shifting of focus led a student to produce an articulation of the type E5, that is, an internal articulation that put the explicit under check. The utterance resulting from

this articulation was represented by the flow number 36. All of this related to the task 1. On the other hand, the lighter flows correspond to: (a) the students' actions in relation to task 2 (for example, the flow number 44 that departs from C2 toward E3, and the flow 45 that departs from E3 toward I6); (b) the teacher's utterances (for example, the flow number 18 that departs from I6 towards C7).

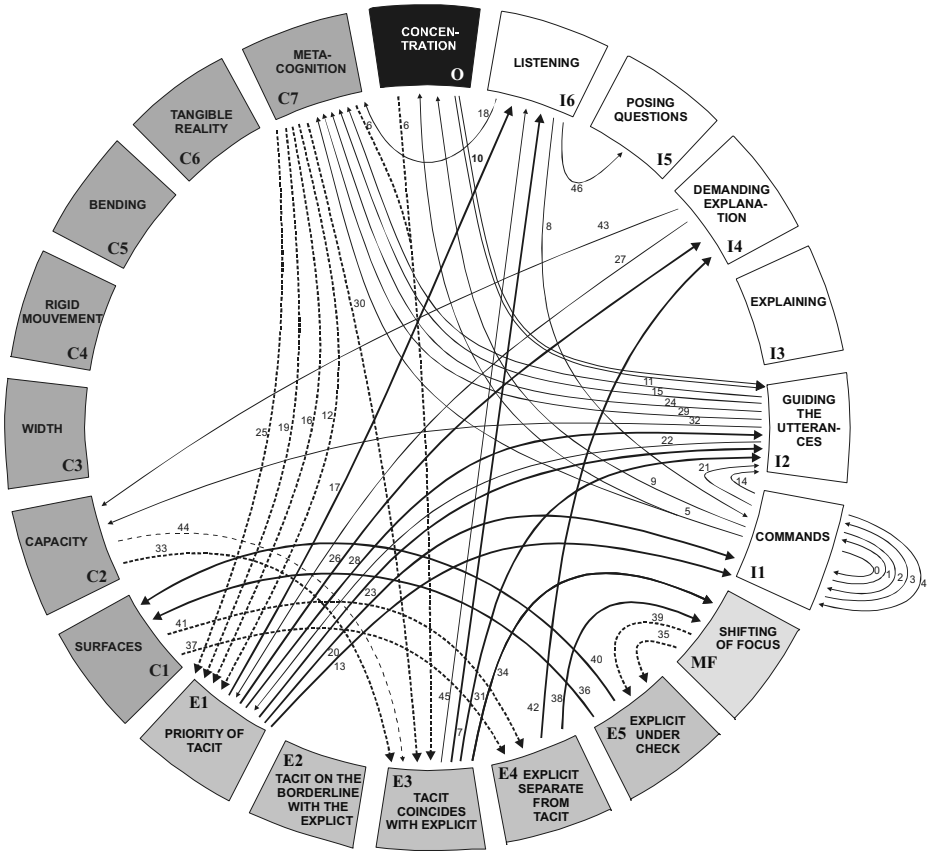

Table 1: Graph 1

The numbers given to the flows correspond to the chronological sequence of the episode in the record's transcription or to the chronological sequence of the meanings produced along the event.

From the observation of the behavior of the flows in each graph, we tried to understand the characteristics of the corresponding segment. When we perceived some regularity or some interesting behavior of the flows (as for example, great

concentration of flows in some specific category), we turned to the record or to its transcription to interpret them.

We would like to observe that students' tacit knowledge about width (C3), rigid movement (C4), bending (C5) and tangible reality (C6) were not identified in segment 1 of the episode. The same can be said in relation to the internal articulation E2, and teacher's intervention of the type I3. That is the reason that we do not find flows departing from those categories. They were identified in subsequent segments whose graphs, due to lack of space, will not presented in this report.

The two main differences between Strom et al's graphs and the graphs in the above-mentioned analysis are: (a) more distance between the observable and non-observable aspects of our categories than in theirs: our inferences were of a higher order, in the sense that they demanded more interpretative effort and riskier evaluation; (b) the flows of our analysis, except number 10, represent more than the chronological sequence of the meanings produced along the event: they correspond precisely to the students' and the teacher's utterances.

FINAL COMMENTS

Concerning the mathematical task under investigation, the methodology demonstrated: (a) the students' knowledge used in a subsidiary way; (b) how the tacit co-operated with the explicit in the projection of that knowledge on the task; (c) evidence of the concentration process (*indwelling*, Polanyi, 1983) shifting of focus and detailing of the particulars of tacit knowledge (Polanyi, 1962, 1983); (d) that, among the various types of tacit knowledge used by the students to accomplish their tasks are the mainly explicit and mainly tacit components of Ernest's (1998b) model of mathematical knowledge.

More precisely, the following components of the model identified were: (1) knowledge related to surfaces of solids: prism, cylinder, hexagon and rectangle, for example, which are included in the mainly explicit component 'statements and propositions'; (2) ontological concept of plane and spatial figures, which is included in the mainly tacit component 'meta-mathematics views'; (3) aspects of oral language, which is a mainly tacit component.

An interesting result that emerged from the analysis is related to the perspective of cognition not necessarily restrict and coincident with language, but seen as a situated social practice, moving between the poles of the tacit – effective action – and the explicit – intersubjective projection of such an action – dimensions. The analysis suggested that student's answer to an oral task may be apparently mistaken under the viewpoint of the discipline. This does not necessarily mean that he/she does not know the correct answer, or else that he/she had not interiorized certain types of knowledge. The presumed mistake or non-interiorization may indicate that, when uttering his/her understanding, the student could be operating either in the ineffable (E1) or in the sophistication domain (E4). If in the former, his/her tacit knowledge was still under construction and therefore predominant over the explicit. This results

in a type of 'painful' utterance, which provides vague clues, thus not enough for us to identify the student's understanding. If in the latter, the tacit functioning of the student's thinking could have been blocked due to some speech ineptitude: the student's symbolic operations seemed not yet ready to express his/her understanding. As a result, his/her utterance is filled with imprecision or contradiction, although produced confidently and with no hesitation.

Exercising her sensibility to understand the student's *modus operandi* through the categories proposed in this analysis can help the teacher identify which stage of learning – tacit prevailing over the explicit, tacit on the borderline of the explicit, tacit and explicit coinciding or tacit and explicit independent – the student is in. Depending on the stage identified, the teacher can create pedagogical supports to promote practices of conversation to help students align, as much as possible, their internal articulation to the domain in which the tacit and the explicit coincide. This is fundamental in the processes of formalization and social communication of the student's mathematical knowledge.

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TRACKING PRIMARY STUDENTS' UNDERSTANDING OF THE EQUALITY SIGN

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Recent curricular reforms are following the lead of the Standards 2000 and, in diverse ways, integrating some algebraic work into primary school mathematics. Our research aims at producing a tool that will allow researchers to track the development of algebraic thinking in children as they progress through various primary programs. In this paper we will illustrate how we have proceeded and what we have learned in our exploration of one small but widely recognized element of algebraic thinking, a rich understanding of the equal sign. An analysis of questionnaires administered to a kindergarten, a grade three and a grade six class in a Montreal area school allows us to present a wide range of student errors and to suggest some key questions for tracking students' thinking and for comparing them across curricula.

INTRODUCTION

Over a decade of research into students' understanding of introductory algebra (Hart, 1981; Booth, 1984, 1988; Wheeler & Lee, 1987; Kieran, 1992; etc.) confirmed what was already popularly recognized[1]: algebra constitutes a major obstacle for a significant number of middle and high school learners. There have been a large number of responses to this problem which have mainly centered on reforming the teaching of high school algebra: problem solving approaches, strengthening arithmetic skills, slowing down the introduction of algebra, various technological approaches. One response—by a growing group of researchers and considered unthinkable two decades ago[2]—has recently found expression in a large number of curricular reforms in North America. The American *Standards 2000* has enshrined the response of the early introduction of algebra by creating an "algebra strand" running in parallel to the traditional arithmetic and geometry from kindergarten onwards.

Many primary curriculum reforms have followed suit. In Quebec[3], where the primary reform is now in its third year of implementation, the development of algebraic thinking is present as an aim though always hidden. The proposed high school curriculum admits that the student has already been introduced to algebra in primary

[1] This opacity of algebra found expression in jokes and phrases such as the French "C'est de l'algèbre" [It's algebra] for the English phrase "It's Greek to me".
[2] In 1984, an ICME-5 working group softened its name from "Algebra in Elementary School" to "Algebraic thinking in the early grades" because, as Davis (1985) expressed it, some participants "were opposed to the whole idea" (p. 198).
[3] The Quebec reform, which is much more timid in its recommendations concerning algebra, appears to have been inspired by the 1989 Standards and the 1994 Vision of Algebra document.

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school but "à son insu" [without his being aware of it]. The Canadian Atlantic provinces are much more explicit about the inclusion of algebra in their K-8 curricular agreement of the year 2000. The document, *Mathematics Learning Results* (Education Foundation of the Atlantic provinces, 2000), mentions algebra in the formulation of all four of its didactic learning principles and puts "patterns and relationships" as one of the four content domains.[4]

The task of creating appropriate didactical tools for the teaching of algebra in the early grades—textbooks, inservice and preservice teacher training programs, evaluation tools—has barely begun. NCTM has suggested a few classroom activities in its *Navigating through algebra* (2001) series. Carpenter, Franke and Levi (2003) have recently produced a textbook for use in teacher training, *Thinking mathematically: Integrating arithmetic and algebra in elementary school.* An analysis of reform textbooks in Quebec indicates that most all but bypass the algebraic goal[5].

Our research falls within the "evaluation tools" category; it aims at tracking students' algebraic thinking throughout primary school. This tool, intended chiefly for researchers, will allow us to compare various primary school programs—those that aim at the introduction of algebraic thinking to varying degrees, those that build algebra on arithmetic, those that develop algebra independently or in relation to the entire curriculum and those that begin school with algebra and build arithmetic on a basis of algebra[6]—and to trace the development of algebraic thought in individual students as they move through the grades within these programs.

In this paper, we will look at one small but widely recognized element of algebraic thinking, the concept of equality and the related understanding of the equal sign, as an example of the construction and use of this instrument. Many researchers have concluded that one of the requirements of the passage into algebra is a much richer understanding of the equal sign than that which is provided by traditional arithmetic and most traditional algebra curricula deal with this early on or relegate it to what they call pre-algebra.[7]

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[4] The others are: number and operations, shapes and space, data analysis and probability.

[5] Indeed, students in the master's program at the Université du Québec à Montréal, concluded that it would be entirely conceivable that a classroom teacher might not even notice the new algebraic aim, so well is it hidden in the curriculum and textbooks. The one textbook series (Défi Mathématiques) for primary school that went furthest in developing algebraic thinking before the reform, has scaled back the algebraic component, apparently to meet the acceptance requirements of the ministry!

[6] The Hawaiian "Measure Up" program is currently adapting and testing the Russian Elkonin-Davydov curriculum in two of its schools. Children begin by comparing and operating on quantities before number is introduced. Thus algebraic thinking precedes arithmetic.

[7] In Quebec, for example, the equal sign has been studied in the first year of high school in what is called a "preparation for algebra".

STUDIES OF EQUALITY

The NCTM *Standards* (2000) consider "equality is an important algebraic concept that students must encounter and begin to understand in the lower grades" (p. 94). This recognition of the importance of the equality concept is based on decades of research from Ginsburg (1977) to Falkner, Levi & Carpenter (1999) and, even more recently, Theis (2003) who did his doctoral research on the learning and teaching of equality in the early grades. One of the findings that have stood the test of time and constituted the basis for the reform programs are that children have a "do something" or operational view of the equal sign rather than a relational one (Behr, Erlwanger & Nichols (1980), Kieran (1981)). A second, and very important finding, is that this unhelpful view of the equal sign does not sort itself out over time or with mathematical maturation. In a study of primary school children, "performance did not improve with age" and "in fact, in this sample, results for the sixth grade students were actually slightly worse than the results for students in the earlier grades" (Carpenter et al., 2003, p. 9). Kieran (1981), who looked at understanding across primary, secondary and university students, suggested that understanding of the equal sign did not necessarily improve beyond primary and may be responsible for a number of errors in secondary and post-secondary mathematics.

A third finding concerning the equal sign—less discussed in the literature but of interest to us for reasons that will soon become evident—was noticed and remarked upon by Ginsburg (1977). He saw how children link the equal sign to the operation and cannot envisage the one without the other.[8] Behr et al. (1980) made a similar observation in their research: "There is a strong tendency among all of the children to view the = symbol as being acceptable in a sentence only when one (or more) operation signs (+, -, etc.) precede it" (p. 15). Rather than interpreting the equal sign as a "do something" signal, it is possible that children perceive + = as a single operator symbol where the + indicates the type of operation to be performed.

OUR RESEARCH

Context

In the spring of 2003, we began exploring a number of "early algebra" themes in kindergarten (35 students, ages 5-6), two grade three classes (20 and 11 students, ages 8-9) and a grade six class (23 students, ages 11-12) in a French Montreal area private primary school with a math enrichment program for all students. Our study of children's understanding of the equality sign involved from two to three hours of video-taped classroom time with each group. We began each class with a full group discussion eliciting reactions to a number of statements involving the number 8 such as 8 = 8, 8 = 3 + 5, 3 + 5 = 4 + 4. Students were then asked to complete a two-page questionnaire that involved filling in the missing numbers in statements of the form

[8] For example, children find 3 + 5 and 3 = 3 unacceptable statements. They will insert an equal sign in the first (3 + 5 =) and an addition sign in the latter (such as 0 + 3 = 3).

a = a, c = a + b, a = b + c, a + b = c + d where one of a, b, c or d was missing (a box or a blank) and the numbers involved in the statements were grade appropriate.[9] In the kindergarten class, a second one-hour session a few weeks later allowed us to repeat the activity using concrete material. The full group discussion involved numbers of candies in small opaque boxes rather than numbers on the board. The questionnaire made the link with the boxes of candy through representations of the various shaped boxes sometimes with small circles to represent the candies (Smarties) and sometimes with numbers.

Results

Few errors occurred in problem types a = a and a + b = c. The tables below summarize the errors which occurred in problems of the form c = a + b (Table 1a) and problems of the form a + b = c + d (Tables 1b and 1c).

Grade	Problem	Reflect number	Sum(not difference)	Difference
K	_ = 4 + 3	4 = 4 + 3		1 = 4 + 3
Gr. 3	_ = 45 + 50	45 = 45+50		5 = 45 + 50
Gr. 6	_ = 6500 + 500			6000 = 6500 + 500
K	7 = _ + 1		7 = 8 + 1	
Gr. 3	67 = _ + 67 67 = _ + 50	67 = 67 + 67 67 = 67 + 50		
Gr. 6	67000 = _ + 50		67000 = 67050 + 50	
K	7 = 7 + _ 7 = 5 + _	7 = 7 + 7 7 = 5 + 7	7 = 7 + 14 7 = 5 + 12	
Gr. 3	67 = 5 + _ 67 = 67 + _		67 = 5 + 72 67 = 67 + 134	
Gr. 6	67000 = 5 + _ 67000 = 67000 + _		67000 = 5 + 67005 67000 = 67000 + 134000	

Table 1a: Errors in problems of the form c = a + b

Here we see that children's errors involved 1) reflecting a given number to the other side of the equal sign as in a = a + b, c = c + b or c = a + c, 2) inserting the sum of the two given numbers in the blank independently of the position of the unknown, and 3)

[9] Single digit numbers were used in kindergarten, double digit in grade 3 and up to 5 digits in grade 6.

inserting the difference of the two given numbers in the blank. When three numbers are given and a fourth to be found as in a + b = c + d, behavior was a little more complex. In Table 1b we see that when the blank was in the c position, for example, some children behaved as if d were not there and simply entered the sum of a and b. Others appeared to ignore the a and "complete the sum" b = c + d while others simply repeated one of the terms a, b, or d in the blank.

Grade	Problem	Direct sum	Complete the sum	Repeat one of terms
K	2 + 4=_ + 5 4 + 8=_ + 5 3 + 5=_ + 8	2 + 4 = 6 + 5 4 + 8 = 12 + 5 3 + 5 = 8 + 8	4 + 8 = 3 + 5	2 + 4 = 4 + 5 3 + 5 = 3 + 8
Gr. 3	4 + 8=_ + 5	4 + 8 = 12 + 5	4 + 8 = 3 + 5	
Gr. 6	4 + 8=_ + 5		4 + 8 = 3 + 5	
K	3 + 5 =2 + _			3 + 5 = 2 + 3 3 + 5 = 2 + 5
Gr. 3	36 + 54 =52 +_			36 +54 = 52 + 36 36 +54 = 52 + 54
Gr. 6	36000 + 54000 = 52000 + _			36000 + 54000 = 52000 + 36000
K	2 + _ = 2 + 5 2 + _ = 5 + 2	2 + 7 =2 + 5 2 + 7 = 5 + 2	2 + 0 = 2 + 5 2 + 3 = 5 + 2	2 + 2 = 2 + 5
Gr. 3	25 +_= 25 + 45 25 +_ = 45 + 25		25 + 0 = 25 + 45	

Table 1b: Errors in problems of the form a + b = c + d

Three additional errors were also revealed: filling the blank with the sum of all the given terms (K) and filling the blank with either the sum (Gr. 3) or the difference (Gr. 6) of two of the terms (see Table 1c).

Grade	Sum of all terms	Partial sum	Difference of some terms
K	2 + 4=11 + 5 4 + 8 = 17 + 5 3 + 5 = 2 + 10 2 + 9 = 2 + 5		
Gr. 3		36 + 54 = 52 + 88	25 + 20 = 45 + 25
Gr. 6			36000 + 54000 = 52000 + 16000 36000 + 54000 = 52000 + 18000

Table 1c: Errors in problems of the form a + b = c + d (particular cases)

Analysis of results

The aim of our study is to create an instrument to track algebraic thinking across many different primary curricula and a wide range of students. We are therefore interested in finding the best ways of eliciting this thinking. Using this particular tool of a written questionnaire to look at children's understanding of the equal sign, the focus of our attention is on finding which of the given problems are the most revealing of children's thinking and of its evolution over time. Before discussing that particular question, we will briefly discuss two unexpected but ultimately fruitful results that confirm the need for the instrument we are creating.

If, as the literature suggests, children all have difficulty with the equal sign and appear to regress in their understanding over primary school, then there would be no point in conducting this study or in modifying curricula to deal with this problem. The hope is that an early introduction of equality will change this portrait. Although we did not intend to compare our students with others described in the research, we did include, in all four questionnaires, one very fruitful question that was used by Falkner et al. (1999) in their study across the grades: 8 + 4 = __ + 5. The table below shows the percentage of students getting the correct answer of 7 in the two studies.

Falkner & al. study		Freiman & Lee study	
Grades	% getting answer 7	% getting answer 7	Grades
1 and 2	5	3	K
3 and 4	9	77	3
5 and 6	2	86	6

Table 2: Comparison of results of students on the question 8 + 4 = __ + 5

Although our kindergarten students were comparable with those in the Falkner study, by grades three and six our students were greatly outperforming their American

counterparts and were drastically improving with grade level.[10] Neither study involved children who were exposed to primary curricula aimed at developing algebraic thinking. Our students, though they represented a full spectrum of abilities, did have one major curricular advantage: they were exposed, for several hours a week, to a math enrichment program. Thus it seems that a written questionnaire could be helpful in comparing understanding of the equal sign in different curricula as well as among students following the same curriculum.

This brings us to the question as to what are the most fruitful questions on the equal sign to be included in a questionnaire on algebraic thinking.[11] Among the less fruitful questions are those for which the majority of students were successful. Although statements in the form a + b = c did produce some incorrect answers, from 91 to 100% of students per class were successful. The statement a = a only produced errors in the kindergarten class where nevertheless 25 of the 33 students responded correctly. A third category of less useful questions has nothing to do with question form and more to do with the choice of appropriate numbers for the grade six class. Our use of multi-digit numbers in some of these questions elicited student errors that had little to do with their understanding of the equality symbol and a lot to do with their inability to operate on large numbers. For example, the question 67000 = 5 + __ elicited nine different erroneous responses ranging from 65995 to 68995. Indeed the majority of errors in grade 6 could be attributed to errors in addition and subtraction.

The questions that discriminated the most among our students are candidates for a future more global questionnaire. Two problem types—both involving a blank in the last position—caused consistent difficulties across the grades: c = a + _ and a + b = c + _. Three problem types caused difficulties across two grade levels: _ = a + b and a + b = _ + d (K and Gr. 3), a = _ + b (Gr. 3 and Gr. 6 where math errors in 67000 = _ + 50 accounted for most of the difficulty). Thus, by order of importance in a short list of problems to include in a written test instrument across the grades, we would suggest: a + b = c + _, a + b = _ + d, c = a + _, a + b = _ + d.

We experimented in grades 3 and 6 with problems having two or more blanks such as: _ + a = _ + b, a + _ = e = _ + d, _ + _ + b = _ + a, and a + _ + a = b + _. Of these problems, a + _ + a = b + _ caused some difficulties at both grade levels. More testing is needed of these forms before we can make a decision as to their usefulness.

Future research

We have not discussed here the video taped whole class sessions where the children's comments were quite revealing of their thinking. Nor have we had time to discuss the

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[10] Two of Lee's students (Ouellet & Tremblay) took this question into grades 7 and 8 of a French public school in a different area of the city and, though the results were inferior (ranging from 60 to 66% in the three grade 7 classes and from 74 to 82% in the two grade 8 classes), they were still significantly superior to the Falkner results and did improve from grade 7 to 8.

[11] We are assuming here that a questionnaire on algebraic thinking will cover many other concepts besides equality and will necessarily be of limited length.

kindergarten children's work using the concrete materials and compare that with the results reported here. We are also planning some interviews to further explore some aspects of children's thinking about equality and the equal sign.

A number of other elements of algebraic thinking have also been explored and these will be the subjects of future papers. We have learned a number of lessons in our exploration of algebraic thinking so far and these will be applied to our study of other relevant themes until the research tool for tracking algebraic thinking across primary school is ready for testing in a variety of contexts.

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LEVELS OF STUDENT RESPONSES IN A SPREADSHEET-BASED ENVIRONMENT

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The purpose of this report is to investigate the range of student responses in three domains - hypothesizing, organizing data, and algebraic generalization of patterns during their work on a spreadsheet-based activity. In a wider context, we attempted to investigate students' utilization schemes of spreadsheets in their learning of introductory algebra. Twenty students' responses to an investigative assignment were analyzed. The findings indicate a wide range of student responses. In each of the three domains analyzed, most student responses fell into several clearly definable categories. However, an attempt to establish a hierarchy of performance levels led to less clear results.

Background

Researchers and educators suggest using various models of learning environments, that widen and enrich the scope of learning processes for students having differing mathematical abilities. Technological tools were recognized as a particularly effective means to achieve this purpose (see for example, Balacheff & Kaput, 1996).

The *Compu-Math* learning project created a technologically based learning environment, which systematically covers the entire mathematics syllabus for grades 7-9. As described by Hershkowitz and her colleagues (2002), this project is based on the following principles:

- Investigation of open problem situations;
- Work in small heterogeneous groups, where the problem is investigated and discussed;
- Consolidation of the mathematical concepts and processes, that arose in group work, in a whole-class discussion;
- Investigations that utilize computerized tools to facilitate operations within and between various mathematical representations, to reduce the load of formal algorithmic work, to enable the construction of mathematical concepts and processes, to provide feedback to hypotheses and solution strategies, and to resolve a real need to explain important processes and products;
- Interactions between students in a group or in the class as a whole, between students and computerized tools, and between students and the teacher;
- Reflective actions on the learning process;

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This report investigates the range of student responses in three domains: (1) hypothesizing, (2) organizing data, and (3) algebraic generalization of patterns during students' work in a spreadsheet based activity. The findings address the broader issue of detecting the learning processes underlying introductory algebra students' *instrumental genesis* (Mariotti, 2002). Here we attempted to describe and discuss the processes involving students' perception and utilization of spreadsheets as a technological artifact.

Our choice of domains was guided by their importance in the process of learning mathematics in general, and in the context of spreadsheet-based learning of introductory algebra in particular, and also by their potential to characterize and possibly indicate levels of mathematical learning. The importance of each of the three investigated domains was considered by others on various occasions:

- Chazan and Houde (1989) and Howe and his colleagues (2000) indicate that students' hypothesizing has the potential to be a source of authentic mathematical activity, a catalyst for meta-cognitive action, and a motivator for learning.
- Student performance at the stage of collection and organization of data was investigated in particular in the domain of data analysis (Ben Zvi & Garfield, in press) and spreadsheet-based mathematical activities.
- Algebraic generalization of patterns is considered central for the learning of algebra on the one hand, and as a source of student cognitive difficulties on the other hand (Kieran, 1992).

Methodology

One of the authors (M. T.) was the teacher-researcher of a Grade 7 introductory algebra class consisting of 24 students. For this particular group, all five weekly lessons were conducted in a computer laboratory; students had access to a computer, and occasionally used *Excel* spreadsheets as tools for mathematical work and documentation. In Grades 5 and 6, the students were also occasionally engaged in exploratory mathematical activities that employed *Excel* spreadsheets. During selected lessons, the teacher conducted both video and audio recordings. The video recordings included the class work of one pair of students and discussions involving the entire class. The audiotapes recorded the discussions of 3-4 additional pairs of students. In addition, all the students were required to save their computer work on the net. The data analyzed in this report is based on the data collected during one 90-minute lesson. The data include transcriptions of the video recording of one pair, the audio recordings of four pairs, and 12 spreadsheet files of all students. Twenty students attended this lesson, and worked in 6 pairs, one group of three, and 5 individuals. The lesson analyzed here was based on an activity called *Growing Rectangles*, and was conducted at an early stage of the course – three weeks after the beginning of the school year.

The activity of *Growing Rectangles*

The situation associated with the problem (Figure 1) presents the process of growth of three rectangles; students are requested to relate to the first ten years of this process.

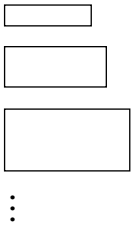
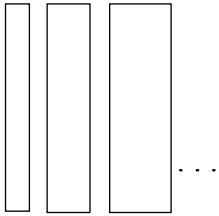
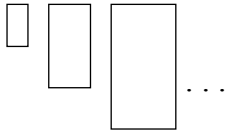
Rectangle A	Rectangle B	Rectangle C
		
At the end of the first year, its width is one unit, and it grows by an additional unit each year.	At the end of the first year, its width is one unit, and it grows by an additional unit each year.	At the end of the first year, its width is one unit, and it grows by an additional unit each year.
The length of this rectangle is always longer than its width by three units.	The length of this rectangle is constantly 10 units.	The length of this rectangle is always twice the length of its width.
At what stages of the first ten years does the area of one rectangle overtake another's area?		

Figure 1. Problem situation of the *Growing Rectangles*.

At the initial stage, the students are required to predict (hypothesize), without performing any calculations or formal mathematical operations, which rectangle will overtake another's area, and at what stage.

Next, the students are required to organize their data regarding the growing rectangles in a spreadsheet table, record the formulas used to construct their table, and compare (first numerically and then graphically) their findings and their predictions.

Student responses

Next, we will attempt to present and analyze some categories of student responses to this activity. As previously mentioned, we will restrict our analysis here to three domains: hypothesizing, organizing data, and algebraic generalization of patterns.

Hypothesizing (predicting results). In this domain, we observed three categories of responses.

- *Local considerations.* Most students sampled one or more points on the time sequence and drew conclusions according to their findings in these selected points. For example, two pairs chose to look at the fifth year (probably because of its being the midpoint of the given period of ten years) and realized that at this point, the areas of Rectangles B and C are equal. This led them to conclude that these

rectangles become equal in area every five years. They rejected this hypothesis at a later stage.

- *Considerations of rate.* A few students considered the rectangles' rate of growth. One pair reasoned as follows: Rectangle B has a fixed length, and as a result, it cannot "win". Between Rectangles A and C, Rectangle C is the "winner", since at each step "it grows by itself".
- *Eliminating the common variable.* One pair noticed that one side of each rectangle is the same at each stage and grows similarly. As a result, they ignored the contribution of this side to the area, and considered only the growth process of the other side. The comparison of the three corresponding sequences of lengths (Rectangle A - 4, 5, 6, …; Rectangle B - 10, 10, 10, …; Rectangle C – 2, 4, 6, …) led them to conclude that Rectangle C will have the largest area at the end of the process.

Ben Zvi and Arcavi (2001) found that student analyses that are based exclusively on local considerations lead to poorer results, as compared to an argumentation that is based on global, or combined global and local considerations. Thus, we considered the first category of hypothesizing to be at a lower level, in comparison to the other two. We could not develop an argument with regard to the comparison of the other two categories. They are both global and based on general features of a variation process. One should note, however, the elegant simplicity of the third strategy.

Organization of data. In this activity, the students were required to use *Excel* in order to collect, organize, and analyze their data. However, they were not instructed *how* to organize their data. Figure 2 presents the four categories of tables observed in the students' work files and their corresponding frequencies.

- *Separate tables*. Students in this category constructed three tables - one table for each rectangle (Figure 2a). Each table contained four columns to describe the year (from 1 to 10), the rectangle's two linear dimensions, and its corresponding area.
- *Extended table* (Figure 2b). The tables of this category contained ten columns (i.e. variables): the year, six columns for the linear dimensions of each rectangle, and three columns for the area measures. The columns were in order either by rectangle (i.e. an annexation of the three separate tables previously described) or by variable (i.e. first, grouping together the linear dimensions of all rectangles and then, their area measures).
- *Reduced table* (Figure 2c). These students allotted only one column for the width measures, since they are identical for all three rectangles. Thus, the number of columns in these tables was reduced to eight.
- *Minimal table* (Figure 2d). These students noticed that the width measures are identical to the year number and omitted the width measures altogether. Moreover, they omitted the length measures as well, and included their corresponding expressions directly into the area formulas of each rectangle. Thus, the number of columns in this case was further reduced to four.

Rect. A	Width	Length	Area
1	1	4	4
2	2	5	10
3	3	6	18
4	4	7	28
5	5	8	40
6	6	9	54
7	7	10	70
8	8	11	88
9	9	12	108
10	10	13	130

Rect. B	Width	Length	Area
1	1	10	10
2	2	10	20
3	3	10	30
4	4	10	40
5	5	10	50
6	6	10	60
7	7	10	70
8	8	10	80
9	9	10	90
10	10	10	100

Rect. C	Width	Length	Area
1	1	2	2
2	2	4	8
3	3	6	18
4	4	8	32
5	5	10	50
6	6	12	72
7	7	14	98
8	8	16	128
9	9	18	162
10	10	20	200

(a) Separate tables (2 files)

Year	Width Rect. A	Length Rect. A	Area Rect. A	Width Rect. B	Length Rect. B	Area Rect. B	Width Rect. C	Length Rect. C	Area Rect. C
1	1	4	4	1	10	10	1	2	2
2	2	5	10	2	10	20	2	4	8
3	3	6	18	3	10	30	3	6	18
4	4	7	28	4	10	40	4	8	32
5	5	8	40	5	10	50	5	10	50
6	6	9	54	6	10	60	6	12	72
7	7	10	70	7	10	70	7	14	98
8	8	11	88	8	10	80	8	16	128
9	9	12	108	9	10	90	9	18	162
10	10	13	130	10	10	100	10	20	200

(b) Extended table (4 files)

(c) Reduced table (2 files)

Year	Width	Length Rect. A	Area Rect. A	Length Rect. B	Area Rect. B	Length Rect. C	Area Rect. C
1	1	4	4	10	10	2	2
2	2	5	10	10	20	4	8
3	3	6	18	10	30	6	18
4	4	7	28	10	40	8	32
5	5	8	40	10	50	10	50
6	6	9	54	10	60	12	72
7	7	10	70	10	70	14	98
8	8	11	88	10	80	16	128
9	9	12	108	10	90	18	162
10	10	13	130	10	100	20	200

(d) Minimal table (4 files)

Year	Area Rect. A	Area Rect. B	Area Rect. C
1	4	10	2
2	10	20	8
3	18	30	18
4	28	40	32
5	40	50	50
6	54	60	72
7	70	70	98
8	88	80	128
9	108	90	162
10	130	100	200

Figure 2. Categories (and frequencies) of tables observed in student *Excel* work files.

First, a distinction can be made between the construction of separate tables and the other categories. Students who employed the first strategy did not consider the common features of the three rectangles and the task at hand. A numerical or graphical comparison of several processes of variation requires either a common table or a common graph. An analysis of the other three table categories led us to conclude that an increasing level of conciseness is related to higher level of reasoning. As indicated by the findings presented in the next section, the construction of a compact table is related to the abilities to detect patterns and to express symbolically the relationships involved in this particular problem situation.

Algebraic generalization of patterns. Hershkowitz and her colleagues (2002) indicate that the use of spreadsheets to investigate processes of variation enables students to use spontaneously algebraic expressions. Spreadsheet users employ formulas (expressed in spreadsheet syntax) as a natural means to construct extensive numerical tables and then, possibly to plot graphs. In our case, after three weeks of learning algebra, all students, with one exception, were able to write and then copy ("drag") spreadsheet formulas, to obtain the necessary numerical data. We investigated whether the formulas used by the students in this case have the potential to indicate levels of student ability to generalize algebraically. After examining students' work in this activity, we formulated the following categories:

- One student *exclusively used numbers* and showed no attempt to generalize, but was still able to construct a graph based on his numerical data.
- *Recursive formulas* express a relationship between two consecutive numbers in a sequence. Figure 3(b) presents an example of using recursive expressions for obtaining the dimensions of a rectangle.
- *Explicit formulas* use the sequence place index as an independent variable. In our case, 3 students (in 3 files) used the year number as an independent variable in their expressions (see Figure 3(a)).

	A	B	C	D
1	Year	Width	Length	Area
2	1	=A2	=A2+3	=A2*(A2+3)

(a) Explicit formulas.

	A	B	C	D
1	Year	Width	Length	Area
2	1	4	1	=A2*B2
3	=A2+1	=B2+1	=C2+1	=A3*B3

(b) Mixed recursive and multivariate formulas.

	A	B	C	D
1	Year	Width	Length	Area
2	1	=A2	=A2+3	=B2*C2

(c) Mixed explicit and multivariate formulas.

Figure 3. Algebraic generalizations.

- *Multivariate formulas* use more than one variable to express a generality. In our case, in 8 (out of 12) files the area of the rectangles was expressed by using the letters corresponding to the length, width or year columns (e.g., = B2*C2). The

variables used in a multivariate formula were originally obtained by a recursive method or by an explicit formula (see Figures 3(b) and (c)).

Recursive formulas can be considered to be the result of a local view of a pattern. In standard algebra, recursive formulas are less effective as a tool for finding a required number in a sequence, or for analyzing and justifying sequence properties. In a spreadsheet environment, these disadvantages are less valid and hence less obvious to students (or researchers). The spreadsheets' dragging ability allows us to obtain a very large quantity of numbers by using *any* kind of formula – including a recursive or a multivariate one. Moreover, the same action of dragging enables students to understand the global aspects of a recursive formula. Recursive formulas have a didactical advantage as well. For example, they are easier to understand and produce, and sometimes their use is the only way that some complex (for example, exponential) functions can be introduced at an early stage.

Multivariate formulas are also frequently considered an obstacle to students' performance in algebra. Lee (1996) states that one of the main difficulties in algebraic modeling is not the construction of a general expression, but the finding of a model that proves to be effective in the solution process. Once again, the difficulty of producing an ineffective model is bypassed by the spreadsheets' ability to accept and handle a considerably wider range of generalizations than with a paper-and-pencil environment. In a spreadsheet environment, students frequently replace a quantity previously expressed as an algebraic expression by a new variable. Jensen and Wagner (1981) consider students' ability to view expressions as entities a characteristic of algebraic expert thinking. The contribution of this strategy to advance this skill needs further inquiry.

Summary

Our analysis of student responses in this spreadsheet activity revealed a wide range of student responses. Because of the variety of student responses detected in our findings, we concluded that a spreadsheet-based learning environment enables students to follow different paths of *instrumental genesis*, according to their algebraic reasoning and their perception of the employed artifact.

In addition, we attempted to create categories of responses with regard to students' ability to hypothesize, to organize data and to generalize. In each of these three domains, most student responses could be categorized in several distinctive groups. However, an attempt to distinguish levels of performance among these categories led us to less clear results. In our case, the process of hypothesizing did not require the employed technological tool. As a result, we established levels of performance by an analysis of student mathematical reasoning.

The activity presented here required students to organize large quantities of numerical data. Spreadsheets are particularly well-suited to facilitate the construction of tables. Our findings indicate that this feature enables students of all

levels to organize their data. Moreover, we distinguished various levels of performance in this domain, based on students' level of mathematical understanding of the task.

With regard to students' algebraic generalizations, we found that the spreadsheets' powerful mathematical capabilities enable students to obtain the required results by employing strategies that are considered ineffective in a paper-and–pencil environment. As a result, we could not establish a hierarchy of generalization skills that would be valid for both environments. We also recommend that the effect of work with spreadsheets on students' ability to generalize algebraically in both environments be investigated.

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SENSITIVITY FOR THE COMPLEXITY OF PROBLEM ORIENTED MATHEMATICS INSTRUCTION – A CHALLENGE TO TEACHER EDUCATION

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Teaching can be understood as acting and deciding in a complex system. On that problem oriented mathematics instruction (POMI) can be characterized as very high complex particularly regarding to mathematical-cognitive aspects. To cope with resulting demands in the long run, the teacher has to be sensitive for this complexity. But what does this mean? How can you get some clues for one's degree of sensitivity for complexity of POMI? Is it possible to sensitize teachers or teacher students for this complexity?

POMI AS A HIGH COMPLEX SYSTEM

Requests for a stronger problem orientation in mathematics instruction (e.g. NCTM 2000) has been raised worldwide for a long time. However, international investigations indicated a quite resistance of reality of mathematics instruction particularly to this requests. There are different reasons for this and many constraints are often quoted by teachers (e.g. Zimmermann 1991). In my opinion *one* reason are additional demands of POMI on the teacher concerning mathematical aspects and in particular designing and implementing a corresponding lesson plan.

In didactic, pedagogic and psychological literature teaching is very often described as a complex system (e.g. Arends 1997, Bromme 1992, Clark & Peterson 1986, Davis & Simmt 2003, Kießwetter 1994, Lambert, Loewenberg Ball 1998, Leinhardt, Greeno 1986; Fritzlar 2004 for more references). This view should be specified. A possible and suitable specification can be found in topical research work of cognitive psychologists (e.g. Dörner, Kreuzig, Reither & Stäudel 1983, see also Frensch & Funke 1995). They depict systems or embedded problems on several dimensions, and elementary and complex systems or problems are on the opposite ends of the scales. Partially based on the initial research work of the German psychologist Dörner the dimensions "comprehensiveness", "connectivity", "dynamic" and "low transparency" are often used (e.g. Kotkamp 1999):

Comprehensiveness: This dimension represents the quantity of information to be considered for an appropriate work on the problem. Therefore the extent of comprehensiveness also depends on the particular agent and his model of the situation. If the quantity of information goes beyond the processing capacity of the agent, he must try to reduce comprehensiveness (in an appropriate way).

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Connectivity: This dimension represents the amount of change of system elements on account of modification of other elements. It describes, how close connected the system is. Because of the connectivity of a system, an agent cannot do one thing without many others. That's why he has to consider side and long term effects of his decisions too.
Dynamic: This dimension represents to what extent system elements change without intervention from outside. Dynamic systems do not wait for the agent, and so he often feels time pressure. In addition in dynamic systems an agent has not only to consider the present state, but its development in time too.
Low transparency: This dimension represents to what extent states of the system, its elements and connections between them can be observed. If necessary the agent has to complement his knowledge through active information gathering.
So complex systems can be characterized as comprehensive, close connected, high dynamic and hardly transparent systems, and there is absolutely no doubt that teaching is complex in this sense.

POMI is complex like conventional instruction in regard to the so called "classroom management" (Doyle 1986). But beyond this it is characterized by an additional high complexity concerning math-cognitive aspects: Several, partly different nevertheless coinciding problem solving processes of pupils appear during a lesson, which should be watched and supported by the teacher if necessary. This processes are influenced by numerous anthropogenic or socio-cultural conditions and especially by own, sometimes inconspicuous teaching-decisions in many ways. So a huge network of interactions between conditions, decisions and features of the lesson emerges, which can hardly be overlooked. From there unexpected matters occur very often and the teacher has to free oneself of or at least question own views. In addition, these processes are high dynamic, normally a teacher has only few seconds to find a suitable reaction. So he cannot take into account all available information or exhaust his whole knowledge. Teaching orientated to independent problem solving processes is very low transparent because the teacher cannot look into pupils' minds. In addition he is no longer the only (and authoritarian) source of information and he has to give the pupils more scope for doing mathematics. So in some ways POMI can hardly be under control or be planned.

POSSIBLE CONCLUSIONS

Complexity of teaching outlined above may be hardly disputed, but in my opinion it is considered too little or not the right way in teacher education. Especially complexity of POMI ought to be made a subject of discussion also in math teacher education at university, and – as a first step – educators ought to try to sensitize teacher students in this direction. (In this way teacher education could also make a further contribution to a stronger problem orientation of mathematics instruction.)

But to my knowledge no experiences concerning such orientation of teacher education exist. From there groundwork has to be done first!
Important conditions for a corresponding supplementation of teacher education are

- some information about students' initial situation regarding sensitivity for complexity, and
- possibilities to evaluate new elements of education to what extent they can help sensitize teacher students in this direction.

So within my research I created a diagnostic instrument and tested in a first explorative study, to what extent it can provide clues for the degree of sensitivity of teacher students for the complexity of math-cognitive aspects of POMI. For this I designate (in sense of a provisional definition for working) an agent as sensitive in this regard, if he is aware of the complexity of POMI, of special demands arising from it and of limits of his possibilities to decide and to act in an appropriate way.

Main element of the created diagnostic instrument is an interactive realistic computer scenario, which models selected aspects of the complexity with an appropriate example. In the following I want to report some details about the subject of the scenario, its important features, about the explorative study and main results.

AN ATTEMPT TO ANALYZE SENSITIVITY FOR COMPLEXITY

A mathematical problem – the "Faltproblem"

Subject of the computer scenario is the use of the following problem in a math lesson:

> The *Faltproblem* (folding paper-problem): A sheet of usual rectangular typing paper is halved by folding it parallel to the shorter edge. The resulting double sheet can be halved again by folding parallel to the shorter edge and so on.
> After n foldings the corners of the resulting stack of paper sheets are cut off. By opening the paper, it will be detectable that (for $n>1$) a mat with holes has resulted.
> Find out and explain a connection between the number n of foldings and the number $A(n)$ of folding-cutting-operations.[1]

Teachers, students and the author tried out this problem in about 50 lessons mainly in fourth and fifth grades of different school types. This experiences showed special potentials of the *Faltproblem* for POMI, which can only be listed here: the problem can be understood very easily (also by young pupils), nevertheless it is not at all mathematical simple; many possibilities to come to terms allow a differentiated work on the problem; very often pupils can evolve presumptions, search for explanations and, more generally, work heuristically; the problem is a motivating challenge for pupils,

[1] Used formulation of the problem and goal of working on are intended for the teacher. This problem was developed by Kießwetter (e.g. Kießwetter & Nolte 1996) to use in an entrance examination of the university of Hamburg.

generally they enjoy working on it; there are many possibilities for communication and cooperation; the problem is open in regard to ways and also to goals of working on; it has many points of contact to several mathematical subject areas and other mathematical problems; and many variations and extensions are possible.[2]

But the use of these potentials leads to a higher complexity and to additional demands on the teacher like outlined above.

The computer scenario

By analyzing the lessons about the *Faltproblem* I collected important aspects of the underlying network of conditions, decisions and features of pupils' problem solving processes. Based on this, on interviews of students, teachers and teacher educators and on theoretical literature I created a descriptive model and implemented it in an interactive computer scenario, which confronts the user with decision-situations connected to the use of the *Faltproblem* in a fifth grade's lesson. In this scenario the *Faltproblem* can be virtually taught in three different classes. For that the user takes the part of the teacher. At first he can decide about the lesson goals and how to begin the work on the problem. Then the scenario models some possible and probable reactions of the pupils, especially their working processes, ideas and results, and the user has to react again. But he can also go back and correct his former decisions or give some additional alternatives for reaction.[3] At the end of the lesson the user is told an assessment of his decisions particularly with regard to his lesson goals. From here he can also go back to former decision-situations or start again. (The figure on the next page illustrates these possibilities of interaction with the scenario.)

The scenario makes it possible for the user to vary his decisions systematically in the same class or to check effects of his decisions in different classes. So he can explore a large number of possible and probable lesson courses. Especially in this combination of realism, interactivity and the intimated possibilities of investigating the scenario I see its potentials regarding to the research goals:

- By realistic modeling of appropriate decision-situations through the scenario some special demands of POMI on the teacher can be simulated.
- A scenario gives a chance to sufficient complex modeling with concentrating on very important but often more or less ignored math-cognitive aspects.
- A scenario enables an interactive investigation. In this way arises a complex network of decision-situations comparable to real teaching.
- Decision-situations can be explored repeatedly (as often as the user want) and without time pressure. In addition for the user it doesn't make any difference, if

[2] For more details see FRITZLAR (2004). I want express my thanks to all pupils, students, teachers and school administrators involved in the investigations.

[3] The scenario cannot react on alternatives given by users. But they were automatically collected and can be used for further development of the scenario.

he is able to execute his plans. (This could be important specially for teaching novices.) Altogether a scenario can model decision-situations realistically, and it enables ways of analysis of these situations, which do not exist in reality but whose use can provide some clues for the degree of sensitivity for complexity.

- Modeled situations can be varied systematically. By this the user can experience complexity of teaching in a special way and the teacher educator can analyze his examination of this complexity.
- As many students as wanted can work with the scenario, and it can be handled in an easy way.

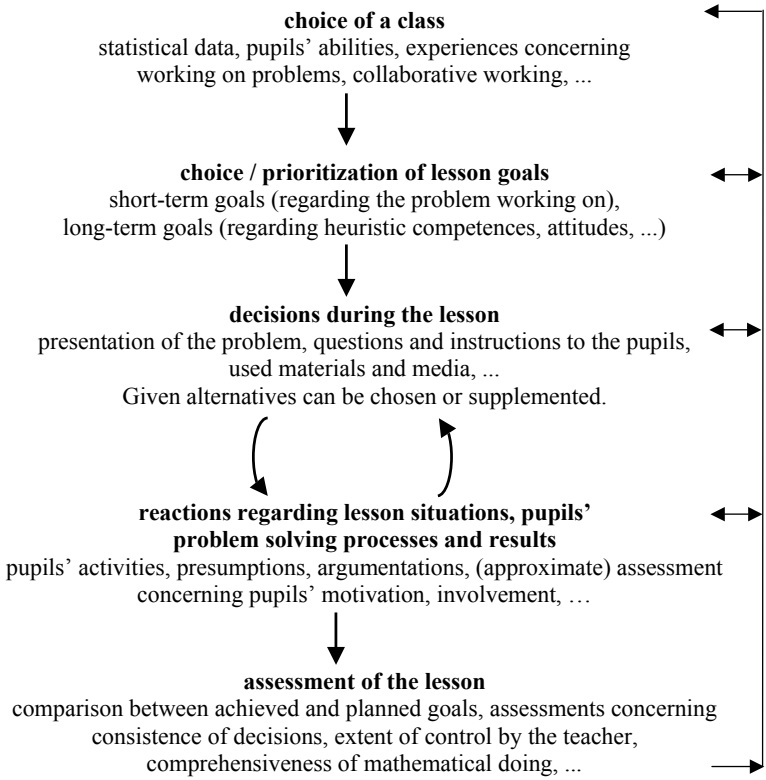

Figure 1: Work on the computer scenario

An explorative study

Within an explorative study I investigated how teacher students work with the scenario (without extensive instructions) to get some clues for their degree of sensitivity for complexity. Therefore I needed some indications for sensitivity, which could only

be partially deduced from preliminary theoretical considerations. On account of the innovative character of the study I had to get indications from empirical data (characteristic features of students' investigation of the scenario) too. Based on these I created a fourdimensional "sensitivity for complexity (sfc) – vector", which describes the investigation of the scenario by the user and with it gives some clues for his degree of sensitivity for complexity. The "sfc – vector" has the following components:

***Exploratory behavior**:* A sensitive agent is expected to try to get to know an unknown system, to come off own conceptions, to scrutinize and to correct own decisions if necessary. Therefore the exploratory behavior of the user can indicate to his degree of sensitivity for complexity, and consequently this component represents quantitative aspects (number of loops and jumps back within the program) and qualitative aspects (e.g. number of different modes of representation the problem) of exploring the scenario by the user.[4]

Context sensitivity: A sensitive agent is expected to analyze modeled situations in a detailed manner and to check offered alternatives for possible (side and long-term) effects. He is expected to consider (detailed modeled) math-cognitive aspects in particular, because these are so very important for POMI. Consequently this component represents to what extent the user referred in decision-situations to problem solving processes of pupils, aspects of the mathematical content, or more social aspects (motivation, teaching methods, …) of the lesson.

***Inconsistence**:* A sensitive agent is expected to react consistently with modeled features of the lesson and (linked to them) his previous decisions. Consequently this component represents the percentage of decisions of the user, which are interpreted to be not consistent with modeled aspects of the lesson, particularly with features of pupils' problem solving processes**.**

Reflectivity: A sensitive agent is expected to try to create an appropriate mental model of connections between different aspects of the lesson. He is expected to question the quality of modeling by the computer program, to create additional alternatives in decision-situations if necessary and to reflect own decisions and his decision behavior. Consequently this component represents the degree of (critical) reflectivity on a rating scale.[5]For more details of the empirical investigation and its results I have to refer to Fritzlar (2004). For a first impression I want to indicate main results and give as examples coarsened results of two "extreme" experimental subjects of the study:

- Differentiated clues for the degree of sensitivity for complexity can be gained by analyzing the investigation of the scenario.
- There are no objections against the independence of the components of the "sfc-vector".
- I could not find specific sensitivity types in the experimental group.

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[4] Of course quantitative and qualitative aspects are not completely independent from each other.
[5] The scale is related to the relative differences between the subjects in this respect.

- Most of the involved students had only a low degree of sensitivity for complexity (regarding to the sfc-components). Generally the scenario was not much explored by the students. Possibilities for systematic testing of teaching-decisions also on different conditions were hardly used. The scenario focuses on math-cognitive aspects of POMI. This was realized by many students, nevertheless features of pupils' problem solving processes were considered only superficially and to a very small extent. Reflectivity of the scenario investigation was low in general. Users hardly reflected on connections between conditions of and decisions during the lesson and pupils' problem solving processes. Also from there arose only few motives for exploration. Multidimensionality of numerous decisions was rarely taken into account, meta-cognition was hardly perceptible.

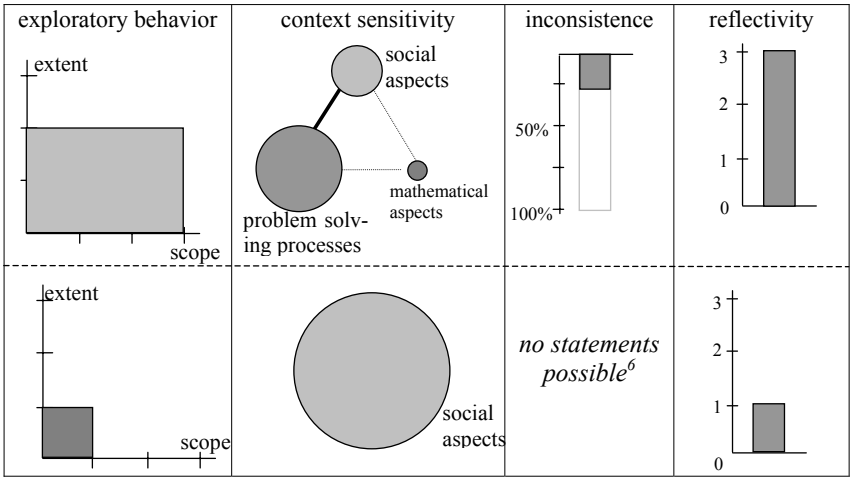

Figure 2: Coarsened results of two teacher students

The study presented here is new in regard to used methods and research goals. That's why I consider it appropriate to work exploratively and related to a concrete example. For several parts of the study almost inevitably arise limits and beginnings for improvements and supplementations. But beyond this the reported computer scenario could also be suitable to contribute to *sensitize* math teacher students for the complexity of POMI. For this I see the following possible potentials of the scenario:

- The program models selected parts of the network of interaction composed from conditions of the lesson, decisions during the lesson and features of pupils' problem solving processes and connected aspects. It can be easily handled and investigated by the user on his own.

[6] The student worked on too few decision-situations.

- Unavoidable reductions were made in view of my goal to enable experiencing specific aspects of the complexity of POMI by working on the program.
- The scenario enables ways of investigation, which do not exist in reality but allow conclusions about it. So it allows experiences, which could hardly or only tediously be gathered otherwise.
- The scenario could widen the "horizon of learning" by confronting the user with possible effects of his decisions directly, hardly delayed and in time-lapse.
- Working on the program could encourage the user to critical reflection of modeled structures and processes, among others in regard to a possible transferability to real situations.

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ICT TOOLS AND STUDENTS' COMPETENCE DEVELOPMENT

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ABSTRACT

In this paper I will present the rationale that motivates the study in an ongoing three-year project following students in school years 8 to 10. The aim is to develop the students' competence with use of ICT tools in mathematics in such a way that they will be able to choose tools for themselves, not rely just on the teacher telling them what to use for a specific task. Experiences and results from research emerging after more than two years will be discussed.

BACKGROUND AND AIM

Based on a constructivist view of learning, the curriculum guidelines for Norway (KUF, 1999), state as an aim that students should develop their knowledge and understanding of the subjects, and independence and self-reliance in their learning. The students should be stimulated to find solutions by explorative, experimental activities, be encouraged to ask questions and investigate different representations and present arguments during their work. Tools like a spreadsheet, a graph plotter and calculators are explicitly mentioned in the curriculum. We find similar recommendations in the NCTM Principles and Standards for school mathematics (NCTM, 2000) and other curriculum plans.

The aims stated in the curriculum guidelines form the background for an ongoing project over three years with students in school years 8 to 10. The students should develop their competence and self-reliance to choose suitable computer tools, not just rely on a teacher telling them what to use for a specific task. The students should learn when and when not to use computer tools, and which ICT tools to use.

The research aims to investigate how students develop their knowledge of the software tools and ability to judge choice of tools and to what extent, if at all, it is possible to achieve the goals in the curriculum guidelines. In this paper I will present the rationale and basic ideas for the project and outline the way we work to achieve our goals including illustrative experiences from classroom practice.

TOOLS FOR MATHEMATICAL TASKS

By ICT tools, or computer tools, in our project, we think of software that makes it possible to use computers to perform tasks that are planned and decided by the user. This means software that is open and flexible, not limited to pre-designed tasks. With such open-ended software tools we can provide learning situations where the students can experiment with mathematical connections, find patterns and be stimulated to develop mathematical concepts and understanding. These can be utilised both to learn mathematics, to learn to use and to choose appropriate ICT tools for problem solving.

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Different kinds of technologies and tools have been used for centuries in mathematics. We can think of tools for measurement, calculations, mathematical notation, symbol systems and written language; cognitive technologies that helps transcend the limitations of the mind (Pea,1987). Computer software is an especially powerful cognitive technology for learning mathematics. This can take the form of an amplifier, which means doing more efficiently the same as before without changing its basic structure. Pea and Dörfler argue we should regard ICT tools as reorganisers. This have wide implications for the objects we work on, in our case mathematical objects, and lead to more activity on a meta level with more emphasis on planning and judging methods (Pea,1987; Dörfler,1993). ICT tools will be a part of the cognitive system: computer visualisations will extend and expand the students' cognition and should be available at any time. This has implications for the kind of software that we chose as tools in mathematics classrooms. Suitable software give opportunity to develop conceptual fluency, provide an environment for exploration and investigation, integrate different representations and stimulate reflection (Hershkowitz et al., 2002).

The CompuMath project which provides long time experience on the use of tools in curriculum development (Hershkowitz et al., 2002), in addition, emphasises the potential of the tool to support mathematization by students working on problem situations and the communicative power of the tool. All the criteria are closely related to the multi-representational nature of the tools, which make it possible to do manipulations of objects and transformations between different representations.

What would then be appropriate tools to use for the students in years 8 to 10 in schools? We think of tools that have more than one kind of use, tools that could be applicable to different content areas, and could be easily available. This means we would look for the generality of the tools, not software produced to cover specific limited areas in the curriculum. A spreadsheet is a good example of such a tool; it is easily available, can be used in many contexts and has become a standard tool in many working places. It can be used to store and analyse data, create number patterns and sequences by general rules and present data graphically.

In the project we use a spreadsheet, Excel, a graph plotter, Grafbox, and dynamic geometry, Cabri. These tools all have the sufficient openness and flexibility to give opportunities for experiments and explorations, which we see as important in developing mathematical concepts according to a constructivist based environment. We also included the use of Internet resources, mainly thinking of collecting information and data to use when that is appropriate.

THE ICT COMPETENCE PROJECT AND METHODOLOGY

Three schools with six teachers and six classes participate in the project with me as researcher. The project leaders, a teacher and I, provide some ideas and material for use in classes, both new and existing material. Software tools, experiences and new ideas have been discussed in project meetings every term. The teachers are

responsible for what and how they implement material in their classes. To some extent the teachers also develop their own material, like prepare spreadsheet models for tasks, and make them available for the rest of the project group.

The schools were selected because we knew they had some experience using computers in mathematics classroom before. But still, some teachers had limited experience and few ideas how to use computer tools in open tasks and admitted they had to learn from others. The project also provides a good opportunity for this.

The activities in the project classes were integrated in the ordinary lessons, and the mathematics teachers are responsible for planning activities.

The project builds on constructivist and social constructivist views of learning and the methodology of the project has aspects of action research and developmental research. Development of teaching ideas for use of the software and support for teachers' competence development was important to support the research and give opportunities to study students' competence development. The research focus implies use of qualitative methods of data collection.

I and a research student visit classes during their work on computers acting as a participating observer and helping teacher. We write field notes and record with audiotape. The teachers write shorts reports from their use of ICT in classes, including what tasks they used and short comments. Students' material in written and on computer files are collected. This material are analysed and compared with observations. In the last part of the project we plan a period of close observation of students' work and interviews with some students of different ability, using audio or videotape. A questionnaire will be given to all students.

As part of the project we investigate methods of analysing the data and categorise outcome concerning students' competence in stages as described in next section. The final evaluation will take place after this paper is written, but some experiences and results can be reported here.

STAGES IN DEVELOPMENT OF COMPETENCE

It is necessary to learn some basic features of the software tools. Insight in what is possible and some fluency with the tools is necessary for students to be able to judge what to use. We need to build this competence over time, and not just leave it to the last part of the students' education. Based on our work in classrooms and recent literature, I, in consultation with the teachers, have designed a developmental framework in three stages.

1. Basic knowledge of the software tools. The students can utilise the functionalities of the software to solve simple tasks prepared for it, when they are told what software to use. For example this could be to make formulas in a spreadsheet when the main outline of the task is given or to use a graph plotter to plot a function when the formula is given.

2. Develop simple models. The students can make the layout of text, numbers and formulas to plan a model for a spreadsheet. For a graph plotter they could judge what functions to draw, use different scales on axis, zoom in or out. Be able to use dynamic geometry to make constructions that can resist dragging, i.e. the figure do not fall apart when parts of it, points or line segments, are moved.
3. Judge the use of tools for a given problem. The students should be able to think of different ways and means for solving a problem, which software is most appropriate to use or when other methods are better.

Development of mathematical competence is involved in all stages. In preparing a formula on a spreadsheet, students gain experiences expressing mathematical connections and further experiments often require use of variables or parameters. In order to develop models or drag resistant geometrical constructions, they have to analyse the situation and build a model according to mathematical rules. The use of computer tools gives the students access to ways of expressing their mathematical models and experimenting with them.

In order to develop their competence the students need experience with different kinds of models and modes of using the software. A variety of applications that use similar patterns can help to form general models for using the software. For example, on a spreadsheet, many problems can be solved using a model similar to a shopping list, the "shopping list model". Another is the "number pattern model": number sequences that can be adjusted using fixed reference to a parameter that is important in the model, and experiment with this. Some basic techniques are necessary, and experience from several applications will help to generalise and build knowledge of models.

Introduction of new tools changes the teaching and learning situation. We can not just introduce a new tool and expect everything to be the same (Pea, 1987; Dörfler, 1993). Good use of the tools implies change in teaching and working style and in the tasks presented to the students. In the next section I will discuss teaching principles we developed and intend to use in the project together with some results.

TEACHING STRATEGIES AND EXPERIENCES

In order to develop competence to choose suitable computer tools, some main points should be emphasised in a teaching strategy. These emerge from analysis of data in the project and from other research dealing with ICT and students' work experimenting and exploring with mathematical connections.

Motivation. A scrutiny of the data analysis suggests that motivation is a crucial point for the students to engage in a problem. We have seen variation in from students' engagement in tasks that they need to see there is a problem of interest. Challenges appear more interesting than routine and tasks with easy solutions. Cognitive conflicting situations and surprising results can be utilised in this connection (Fuglestad, 1998). I observed several times, when students made a figure using Cabri,

and tested by dragging, they had surprises and were challenged to further experiments. There were a lot of discussions and sharing of ideas among students of how to solve the problems. Generally, I observed that many students at the start were motivated to use computers, but this only lasted if the tasks involved also were interesting to them. In a study by Hölzl (2001), presenting an open task, with opportunities for experiments and less obvious answers, the students became strongly involved in investigations. I often observe similar activity with children's learning to use mobile phones with text messages. The high motivation stimulate them overcome the technical difficulties.

Basic features and step by step. It is necessary to know basic features of the software in order to utilise tools. These can be taught in connection with interesting tasks. Our experiences supports the idea to build knowledge from the simple models, step by step without jumping to more sophisticated solutions before the students have experience to understand. For example instead of building a formula with several operations, we have introduced more columns in a spreadsheet to tackle the challenge. Interesting problems can be faced even with limited facilities of the software. We need tasks designed to cover the most common models used for example in a spreadsheet, and the most common properties of geometrical constructions in dynamic geometry.

The topic of some lessons was economics, calculations of salary and taxes. The teacher prepared a file to load into Excel with the framework given and necessary information about the rules for calculations. The students' task was to understand the rules and complete the model by putting in the correct formulas. The tasks aimed at giving some experience using a model and making formulas on a spreadsheet, learning how to use a spreadsheet, and in this way covering parts of necessary basics of a spreadsheet. At first is seemed that most students worked well and some of them were able to make the formulas they needed. However, when I observed closer and discussed with some students, I discovered that the model was complicated and they needed help to develop the formulas. The main problem was that more than one operation was necessary in order to prepare the formula in a given cell. I would suggest giving more intermediate steps in some calculations. The model itself seemed to be more complicated than using the spreadsheet for these students. The task also revealed that some students had problems of understanding how to calculate percentages and proportions of an amount. Similar experiences were observed in other lessons and in my previous research and support the teaching strategy of building form simple models step by step.

Same problem, different tools and methods. Different tools can be used for a problem and different methods using the same tool gives the opportunity to judge and discuss what would be a good solution. In this way students can learn about the suitability of different tools or consider the alternative using just paper and pencil. From an analysis of computer files from students' work on Cabri, I found very different solutions to make the same figure. The students were requested to write a description of their methods, so we could understand their thoughts. We also

observed that students engaged a lot in discussions about their solutions and shared ideas. Using different tools for the same problems have been less explored in the project, but some ideas were developed.

Limited knowledge concerning students' choice of tools is perhaps due to the fact that the tasks are often presented together with the representations and tools that should be employed (Friedlander & Stein, 2001). This seems to be the case in many classrooms, and can be confirmed by looking into school textbooks, official examination tasks and other material. Different methods supplement each other and show a variety of models and representations of the concepts. Students own questions gave good starting points for problems to explore, with computer software as a powerful tool to make sense of information and examine different approaches to problems (Moreno-Armella & Santos Trigo, 2001). In their study a problem posed by a student, was solved in three different ways using computer tools, and later discussed in the class.

Themes and open tasks. We have used open tasks that could be interpreted and solved in different ways with different tools to give the students the option to choose and try out different tools.

With EURO in the pocket, planning a journey was the theme of work in several lessons in two classes. The students had to plan a journey through five countries in Europe, with a commission to accomplish, but with no specific tasks. They had to set the tasks like to convert between currencies, make budgets, make travel plans using a map and timetables and other tasks related to their specific task description. The information could be found on the Internet and they could choose a spreadsheet to prepare tables and make calculations. Observations revealed that some students had problems getting started. After some lessons the teachers reported good activity, and were quite satisfied with the results from some of the groups. Generally the task appeared too open at this stage, giving limited help to get started. The students needed to get used to this way of working and discover the mathematics involved.

Another class had visited a local chemical factory as part of a project work over some weeks. Back in class the students were given four pages with some data and information about the production and Internet links to resources about the factory. There were no questions given, just the challenges for the students to set their own tasks and use the data and material provided, and with encouragement explore what they can do with given data using computer tools. I observed their work and data files were collected and later analysed to look for students' solution methods.

I observed a good working climate and good motivation. Students made tasks on different levels of difficulty, with the possibility to extend the tasks for students who could manage. The observations revealed a variety of tasks and at different levels and some tasks were quite demanding. In some cases students looked at their peers to get ideas, and were challenged by what other students suggested. They shared their ideas and discussed solutions. A similar working method have been used in this class both

before and after, with similar experiences. In a local examination set by the class teacher two weeks later the students were given tasks from the same material and some simple geometrical figures to draw. I observed the students explained their work and found to a large extent they were able to explain their ideas and results. They also discovered their mistakes and seemed to learn from explaining their results.

Reflection and discussion. We found that reflection and discussion are necessary in order consolidate and be sure the students understand the main points in question. In this connection students were be requested to write down their hypothesis before they start exploring patterns and connections and report findings and new questions. In work with Cabri they were requested to describe their constructions. Reflection does not usually occur spontaneously but have to be initiated and writing reports give good help in this connection (Hershkowitz et al., 2002). Drawing conclusions and revising the results are important to get the full value out of the work.

Teachers' intervention: We found in our research, although the teachers had some experience, they expressed they needed help to develop their competence about the software and of how to design teaching modules and tasks utilising ICT tools. In particular, an extra course was set up, and project meeting we discussed the implementation in classes. Developing teachers' competence became an important part, and I think we need more development of this in further work. The teachers' role changes with the introduction of ICT tools and their influence can be crucial at some critical stages in the lessons. Introduction and motivation with examples and challenges at the start and summary and reflection at the end is necessary, not just presenting the solution, but drawing on students' work to reflect over the hypothesis and results (Hershkowitz et al., 2002). During the work, at some important steps, the teacher might intervene, ask questions and point to certain examples to try. Students may not discover important cases to try and in such cases an extra question or suggestion could be the clue to further discoveries e.g. (Fuglestad, 1998). In my observation and interaction with students, I often experienced there was need for just small hints for them to go on in their work, not to give answers but ideas how they could find out themselves. To develop competence to choose tools, the students need to have the option and challenge for that. The presentation of problems and representations of mathematical concepts and choice of tools have deep influence on students' concept development and learning, and implications for later use of ICT tools (Kendal & Stacey, 2003). During the observations and from teachers' report, we had some cases where students also clearly expressed their choice or preference of a tool over another, but so far not many cases.

CONCLUSION

Can the kind of tasks and working methods presented here help students later to judge ICT tools as an option for their problem solving? If at all, to what extent is it possible to see that students achieve a competence to use ICT tools and decide which one is suitable for a particular mathematical problem. We found the description

giving stages in development, although fairly rough; can help to judge the results. At this stage, a few months before finishing their year 10, most students' competence can be described like stage 2 and a few on stage 3.

From the observations so far, and teachers' comments, there is still more to learn to develop students' independence and self-reliance in these matters. In particular, in a new project we have to focus more teachers own competence development.

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THE EFFECTS OF DIFFERENT MODES OF REPRESENTATION ON MATHEMATICAL PROBLEM SOLVING[1]

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The main objective of this study is to investigate the role of four different modes of representation in mathematical problem solving (MPS), and more specifically to develop a model, which provides information about the effects of these representations in the solution procedures of one-step problems of additive structures. Data were collected from 1447 pupils in Grades 1, 2 and 3 of elementary school in Cyprus. Confirmatory factor analyses (CFA) affirmed the existence of four first-order representation-specific factors indicating the differential effects of each particular type of representation and a second-order factor representing the general ability to solve mathematical problems. Results provided support for the invariance of this structure across the three groups of pupils.

INTRODUCTION

There is strong support in the mathematics education community that students can grasp the meaning of mathematical concepts by experiencing multiple mathematical representations (e.g., Sierpinska, 1992; Lesh, Behr, & Post, 1987). Furthermore, Principles and Standards for School Mathematics (NCTM, 2000) include a standard referring exclusively to representations and stress the importance of the use of multiple representations in mathematics learning. The present study purports to throw some light about the nature and the contribution of three systems of representations, pictures, number line and verbal description (written text) of the problem to MPS.

THEORETICAL FRAMEWORK

A representation is defined as any configuration of characters, images, concrete objects etc., that can symbolize or "represent" something else (Kaput 1985; Goldin, 1998; DeWindt-King, & Goldin, 2003). In elementary mathematics teaching and curriculum design, a representation that plays an important role in the teaching of basic whole number operations, and generally in arithmetic, is the number line (Klein, Beishuisen, & Treffers, 1998). Despite the widespread use of number line diagrams as an aid to whole number addition and subtraction, doubts about the appropriateness of using them have been raised (Hart, 1981). Ernest (1985) supports that there can be a mismatch between students' understanding of whole number addition and their understanding of the number line model of this operation. In fact, number line constitutes a geometrical model which involves a continuous interchange between a geometrical and an arithmetic representation. Based on the geometric

[1]This paper is based upon a research project investigating the role of representations in grades 1, 2 and 3 of elementary school, which is supported by the Programme for the Support of Young Researchers, funded by the Research Promotion Foundation in Cyprus.

dimension, the numbers depicted in the line correspond to vectors. According to the arithmetic dimension, the number corresponds to a point on the line. The simultaneous presence of these two conceptualizations may limit the effectiveness of the number line and thus hinder the performance of pupils in arithmetical tasks (Gagatsis, Shiakalli, & Panaoura, 2003).

Although, the mental processes, and particularly the visual-spatial images, used in MPS or mathematics learning have received extensive research in the field of mathematics education (e.g., Presmeg, 1992; Gusev, & Safuanov, 2003), the role of pictorial representations or number line in MPS, has received much less attention. Based on the functions that pictures serve in text processing, as proposed by Carney and Levin (2002) (decorative, representational, organizational, interpretational and transformational), the present study attempted to examine the role of two divergent categories of pictures, decorative and informational pictures, in MPS. Decorative pictures do not provide any information to the pupils for the solution of the problem, but simply decorate the page bearing little or no relationship to the problem context. Informational pictures provide information that is essential for the solution of the problem; the problem is based on the picture.

What is new in this study is that, besides the effect of pictorial representations, it aims to provide information about the effect of the use of number line on problem solving, and compare these effects between each other, and with the effect of the verbal description of the problem. Further, as concerns the use of representations in MPS within the present study, we assume that a major distinction is needed, between auxiliary and autonomous representations. Auxiliary representations are not necessary for the solution of the problem but may assist the process of MPS. Autonomous representations have an essential role in MPS, since, through them, any information related to the problem can be expressed. In terms of this study, we theorize that the number line and the decorative picture are auxiliary representations, while the verbal description (written text) of the problem and the informational picture are autonomous representations.

The present study focuses on one class of problems with additive structures, based on the classification of additive problems, proposed by Vergnaud (1995). In particular, we used one-step change (measure-transformation-measure) problems. We have included problems with positive (join situation) and negative transformation (separate situation) and problems with the placement of the unknown in the starting amount (a) and the transformation (b), that is four situations in total (2x2).

To sum up, the purpose of the present study is to explore and compare the effects of decorative, informational pictures, number line and verbal description (text) of the problem on MPS by pupils of Grades 1, 2 and 3 and investigate how these effects vary in pupils of different age. The following research questions were formulated: First, which are the effects of the particular forms of representations (decorative picture, informational picture, number line and verbal description) on pupils' MPS performance? Second, what are the differences between pupils of different age in

regard with the effect of representations on their MPS performance? It should be noted that the questions of the present study are beyond the mean differences and concern with the structure of pupils' MPS within different modes of representations.

METHOD

The instrument used in this study, to collect information for pupils' MPS, involving different modes of representations, was a questionnaire. The questionnaire consisted of 16 one-step change problems with additive structures (a+b=c). The problems were accompanied with or represented in different representational modes. The exact classification of the problems included in the questionnaire and the symbolization used for them in terms of the analysis of the data are provided in Table 1.

	Join situation (b>0)		**Separate situation (b<0)**	
Type of representation	**Placement of the unknown**		**Placement of the unknown**	
	a	***b***	***a***	***b***
Verbal	V10	V20	V6	V17
Decorative picture	D16	D3	D8	D13
Informational picture	I4	I14	I19	I9
Number line	L12	L7	L18	L5

Table 1: Specification Table of the problems included in the questionnaire

The written questionnaire was administered to 1447 pupils of Grades 1, 2 and 3, from 26 elementary schools in Cyprus (479 1st graders, 477 2nd graders and 491 3rd graders). Pupils were ranging in age from 6.5 to 8.6 years at the time of testing. It should be noted that mathematics elementary books in Cyprus include many mathematical activities based on representations (pictures and number line). Pupils were instructed to use the representations, if they believed they could help them resolve the problems. Answers were marked as 0, 1 and 2. Each correct solution procedure (equation or description in words) was marked as 2, each correct answer without equation or explanation as 1, and each wrong answer or solution procedure as 0. Cronbach's alpha coefficients for the test were found to be well above commonly accepted levels (0.9) of reliability.

Multiple methods of analysis were performed, using the collected data, including Gras implicative analysis (Gras, Peter, Briand, & Philippé, 1997) and Rasch model analysis. Since, however, the present study firstly aims at the articulation of a model explaining the effect of the different types of representations used in MPS, data analysis will focus on CFA. Specifically, the data was analysed by using CFA for the total sample and multiple-group analysis for the different groups of pupils, to explore the theoretical assumption that both the semantics of mathematical problems

(additive structures, placement of the unknown) and the different modes of representation involved in the problems (verbal description-text, decorative picture, informational picture and number line) affect MPS. One of the most widely used structural equation modelling (SEM) computer programs EQS (Bentler, 1995) was used to test the proposed models. In order to evaluate the extent to which the data fit the models tested, the chi-square to its degree of freedom ratio (x^2/df), the Comparative Fit Index (CFI), and the Root Mean Square Error of Approximation (RMSEA) were examined. It is generally recognized that observed values for $x^2/df < 2$, for the CFI > .9 and for the RMSEA < .05 are needed to support model fit. Finally, the factor parameter estimates for the model with acceptable fit were examined to help interpret the models and their divergence.

RESULTS

In order to refute the assumption that MPS is influenced only by the semantics of the problem, a first-order model, which presumes that representation is not a factor in MPS, was examined within the SEM framework. This model involved one first-order factor, which associated all of the tasks and could be taken to stand for general ability of MPS. Specifically, the model hypothesized that: (a) responses to the questionnaire could be explained by one first-order factor representing the ability to solve one-step change problems with additive structures; (b) each item would have a nonzero loading on the factor; and (c) measurement errors would be uncorrelated. This model did not have a good fit to the data [x^2 (98) =1210.29; CFI=.86; RMSEA=.09], and therefore, could not be considered appropriate for explaining the ability of MPS.

To verify that apart from the semantics of the problem, the modes of representation within the problem have a major role in MPS, a second order CFA model was designed. Specifically, the model (see Figure 1 for a diagram of this model and Table 1 for information about the tasks for each factor) hypothesized that: (a) responses to the questionnaire could be explained by four first-order factors that would stand for the four types of representational assistance used here, i.e., pupils' abilities in solving problems represented verbally (V), as an informational picture (I), accompanied by a decorative picture (D), and a number line (L) respectively, and one second-order factor, i.e. pupils' general ability to solve one-step change problems of additive structures (MPS); (b) each item would have a nonzero loading on the factor it was designed to measure and zero loadings on all other factors; (c) measurement errors would be uncorrelated (d) covariation among the four first-order factors would be explained fully by their regression on the second order factor. Figure 1 presents the results of the elaborated model, which fits the data reasonably well [x^2 (94) =483.83; CFI=.95; RMSEA=.05], and shows the parameter estimates. By comparing the second order factor model with the first order factor model, a decrease of the RMSEA and an increase of the CFI could be identified (see Table 2). Thus, the second order model is considered more appropriate for interpreting the ability of MPS.

To test for possible differences between the three age groups in the structure described above, multiple-group analysis was applied where the second order model

was fitted separately on each age group. The model was first tested under the assumption that the relations of observed variables to the first-order factors and of the four first-order factors to the second-order factor would be equal across the three age groups. The fit of this model, although acceptable, was not very good [x^2 (32) = 816.72; CFI=.92; RMSEA=.03]. This was due to the fact that some of the equality constraints were found not to hold (especially, those involving the tasks with an informational picture). As a result, the equality constraints were released.

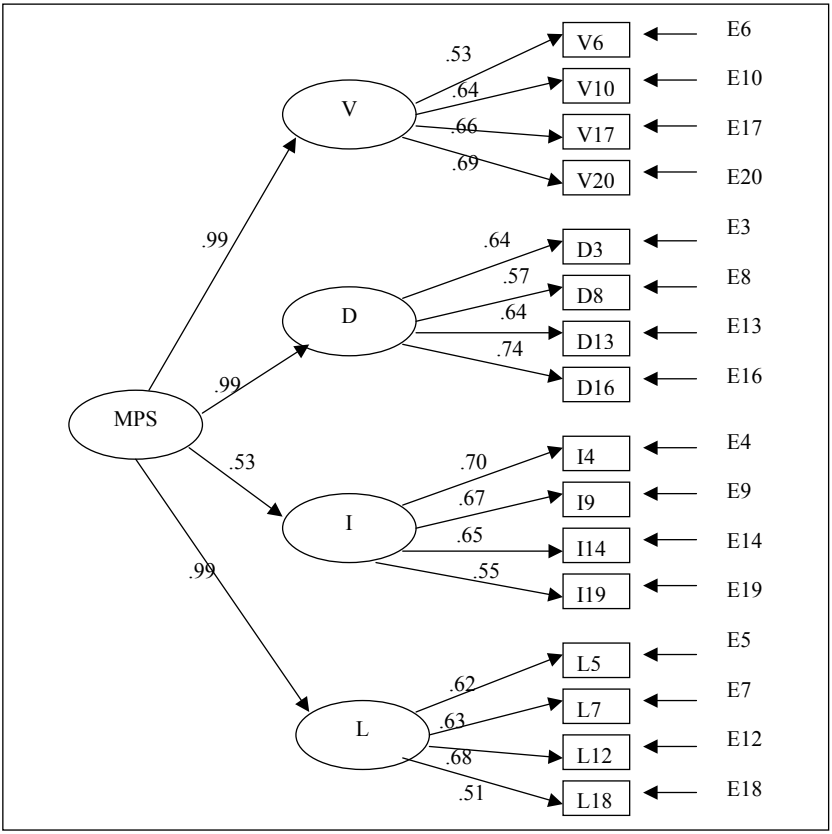

Figure 1: The Elaborated Model for problem solving with different modes of representations, with Factor Loadings and Factor Correlations for all the pupils

Releasing the constraints, a large improvement of the model fit emerged [x^2 (360) = 708.09; CFI = .95; RMSEA = .03] in comparison with the model for the whole sample, as shown in Table 2. In the multi-group model, the parameter estimates for 2nd graders were higher than the estimates for 1st and 3rd graders for almost all tasks.

Moreover, most of the parameter estimates of the model for 1st graders were lower than the estimates of the model for 3rd graders. This finding indicates that the modes of representations involved in the problems did not influence the solution procedures of 3rd graders and 1st graders in the same way as the solution processes of 2nd graders.

Model examined	x^2	df	CFI	RMSEA
General Model with one first order factor	1210.29 p=.0000	98	.86	.09
General model with four first order factors and a second order factor	483.83 p=.0000	94	.95	.05
Multi-group model for 1st, 2nd and 3rd graders	708.09 p=.0000	360	.93	.03

Table 2: Goodness of Fit Indices

Based on the second order models, it is asserted that the first order factor concerning the informational picture is not as closely related to the second order factor as the first order factors involving the other types of representation. This finding suggests that the tasks involving informational picture require extra mental processes relative to the other tasks. The low percentage of pupils (18%) who accomplished a correct solution of the problems involving informational pictures also supports the above result. The high and similar loadings (.99) of the other first order factors on the second order factor reveal that pupils dealt with the problems in verbal form or accompanied with decorative pictures or number line, in a similar and consistent way.

Implicative statistical analysis of the particular data provides support to these findings. In particular, the similarity diagram derived from the implicative analysis, which allowed for the arrangement of tasks into groups according to their homogeneity, contained two basic groups of problems. The group with the greatest similarity consisted of the problems involving informational pictures and the other group contained a combination of the problems with number line, decorative picture and in verbal form. The commonality between the results of both statistical analyses is justified by the distinction between auxiliary and autonomous representations, proposed in our study. Each of the two groups of tasks formed in the analyses corresponds to an autonomous representation, informational picture and text, respectively. The auxiliary representations, which just accompany the problems, function as an adjunct to the verbal description of the problem in the analyses.

DISCUSSION

The main purpose of this study was twofold, to test whether different forms of representation have an effect on MPS and to investigate its factorial structure within the framework of a CFA, across pupils of three different grades. The results provided a strong case for the role of the use of different forms of representations in combination with the semantics of the problems in MPS. The size of the factor coefficients of the proposed model indicates that the ability of pupils to solve one-

step change problems of additive structures is highly associated with the abilities of solving problems in verbal form, with decorative pictures and number lines. The coherence and similarity in the ways pupils handled these representations within their solution processes implies that pupils overlooked the presence of the line or the picture and gave attention only to the text of the problem. This kind of behaviour towards number line can be attributed to the difficulties caused by the mismatch of the conception of number within the context of the problem, as a quantity of items, and the dual conception of number within the framework of the number line, as a vector and as a point (Gagatsis et al., 2003). As regards the decorative picture, the particular finding is in line with Carney and Levin' s (2002) view that decorative pictures do not enhance any understanding or application to the text.

As concerns the function of informational pictures in MPS, from the results of this study, it is evidenced that it differs significantly from the use of other forms of representation in MPS. It was clear that pupils dealt less flexibly with problems involving informational pictures. This indicates that the cognitive demands of the informational picture in the context of mathematical problems are different from the other forms of representations (examined in the present study). These results support Miller's (2000) conclusions that each representational system has its own regularities. Therefore, the results concerning informational pictures and number lines suggest that both kinds of representations need special attention within the context of MPS during instruction. The above findings are in accord with Stylianou's (2001) conclusions that students do not have adequate training associated with the use of visual representations and lack the particular skill.

The stability of the models for the different age groups (1st, 2nd and 3rd graders) provides support for the existence of the same structure of pupils' MPS involving the representational modes explored in this study. However, some differences emerged between the models, as regards the strength of the relationships between the tasks and the factors. An explanation for the 1st graders' lower estimates is that they have not yet developed the mental abilities needed for dealing consistently with the particular forms of representation in problem solving. The decrease of the strength of the relations between the general factors and some of the specialized factors for 3rd graders in comparison with 2nd graders indicates that with development, the functioning of the specialized factors becomes less dependent on general abilities. Thus, 3rd graders used the particular forms of representation in MPS in a more autonomous and flexible manner than 2nd graders. This finding reveals the existence of a possible developmental trend in pupils' abilities in MPS based on different forms of representation. Further research is needed to examine and verify this finding.

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DESCRIPTIONS AND CONFLICTS IN DYNAMIC GEOMETRY

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We expand the theoretical perspective based on the notions of description and conflict, which was previously used to the learning of functions and calculus, to the learning of deductive geometry supported by Dynamic Geometry (DG) environments. Based on prior studies on functions and on the potential role of a DG software, we analyze a case study in which the student's reaction to a problem strongly differ from the pattern observed during in-service courses for teachers. We argue that this student's different background allowed her to experience a conflict, which has given her means to expand her conceptualization. Finally, we contrast this behavior with the narrowing effect of mere formulae application, which has been observed in the typical response given by teachers to the same problem during in-service courses.

INTRODUCTION

We integrate two different aspects of the authors' previous research. In Giraldo & Carvalho (2003a, 2003b) and Giraldo et al. (2003a, 2003b), we addressed potentially positive uses of theoretical-computational conflicts in the teaching and learning of functions and calculus. In Belfort & Guimarães (1998), Belfort et al. (1999) and Belfort et al. (2003), we discussed the role of Dynamic Geometry (DG) environments on the development of deductive reasoning in geometry. The clear indications that conflicts play a crucial role on students' behavior when dealing with DG environments, as also observed by other authors (see Hadas et al., 2000), led us to revisit our work in Dynamic Geometry from a new theoretical perspective. In the following sections, we summarize our previous work, present a mathematical problem explored in DG environment and analyze the reactions of a secondary student when dealing with a conflict situation.

DESCRIPTIONS AND CONFLICTS

In Giraldo & Carvalho (2003b), a *description* is defined to be any reference to a mathematical concept, employed in a pedagogical context, comprising inherent limitations with respect to the associated formal definition. Thus, descriptions may be drawings, formulae, verbal or written sentences as well as computational representations. Limitations of a description may lead learners to situations of apparent contradiction or confusion, when the associated theory seem to have flaws or not to apply to that particular case. We have used the term *conflict* to refer to a situation like this, and the *theoretical-computational conflict* to the case of a conflict associated with a computational description (see Giraldo & Carvalho, 2003a; Giraldo et al., 2003a). Although we have defined a description as a reference comprising

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intrinsic limitations, these limitations may be actualized on the form of conflicts in very diverse manners (if at all). Moreover, the effect of conflicts on learners' concept images (in the sense of Tall & Vinner, 1981; Tall, 1989, 2000) is not a intrinsic feature of the related description, but depends on the whole pedagogical context, including approach adopted by tutors, learners' attitudes and beliefs and so on.

Mathematics education literature provides examples of positive and negative effects of computers' limitations. Hunter et al. (1993) observed that students using software *Derive* did not need to substitute values to get a table and sketch graphs. As a result, they did not develop the skill of evaluating functions by substitution. Even students who could perform the evaluation before the course seemed to have atrophied the skill afterwards. This result uncovers a *narrowing effect*: the intrinsic characteristics of a description lead to limitations on the concept images developed by learners. In our own interpretation, such effect was not due to conflicts, but on the contrary, to their absence (see Giraldo et al. 2003a). Hadas et al. (2000) present a set of DG activities in which the possibility (or impossibility) of a construction was against students' intuition. The activities were meant to motivate the need to prove, by causing surprise or uncertainty. The proportion of deductive explanations was considerably greater in situations involving uncertainty. The authors conclude that uncertainty brought proofs into the realm of students' actual arguments. As a result, they naturally engaged into the mathematical activity of proofing.

In our previous work, we analyzed effects of computational descriptions for the concept of derivative on undergraduate students' concept images. In that case, the limitations were strongly related to the fact that the finite structure of computational algorithms is used to describe a mathematical concept theoretically grounded on an infinite limit process. In Giraldo et al. 2003b, we analyzed six students dealing with a computational description for functions in which a differentiable function seemed not to be, due to graphic windows ranges. We observed that for some students, their previous concept images enabled them to almost immediately solve the theoretical-computational conflict. For others, the conflict motivated the construction of new cognitive units, acting to enrich their concept images. For others yet, no conflict was actualized. In Giraldo et al. 2003a, we analyzed reactions of a student, Antônio (pseudonym), dealing with tasks involving computational descriptions for differentiable and non-differentiable functions, based on the notion of local straightness. We observed that one particular aspect of Antônio's beliefs towards the computer played a crucial role on his behavior and strategies: he spells out clear awareness of the device's limitations and of the possibility of 'mistaken' outcomes to be produced. We observed different effects of theoretical-computational conflicts on Antônio's concept image. In some situations, his previous experience gave him means to quickly grasp the theoretical issues related with the conflicts. In that case, the conflicts acted as a *reinforcement factor*, strengthening his previous knowledge. In other situations, his knowledge of the subject was not enough to enable him to comprehend what was going on; and the conflicts acted as an *expansion factor*, triggering new linkages between cognitive units. In other occasions yet, his previous

beliefs constituted an obstacle to perform the task; and the conflicts acted as a *reconstruction factor*, prompting him to rethink and rebuild structures in his concept image (see also Giraldo & Carvalho 2003a, 2003b).

CONFLICTS IN DYNAMIC GEOMETRY

When we consider the use of Dynamic Geometry, similar results are found in Belfort & Guimarães (1998). We observed in-service teachers' courses dealing with a Dynamic Geometry environment. On one of the activities, teachers were asked to find empirically the rectangle with perimeter $40m$ and greatest area possible. Due to floating point errors, the software could give approximate values. As a result, some of the teachers obtained the maximum area of $100m^2$, but the values for the side were inconsistent. For example, one of them found sides of $10.03m$ and $9.97m$. We reported that, initially, the teachers ended up in a deadlock, and were unable to figure out the correct answer at all. However, we remark that the further investigation about the software 'mistake' led to the necessity to find a theoretical solution for the problem. In fact, similarly to Hadas et al. (2000), we concluded that software's limitations may be used as a powerful tool for the development of deductive reasoning in geometry (see also Belfort et al., 2003).

As discussed in Guimarães et al. (2002), so far, our work on the influence of DG environments was based on previous studies by Parzysz, 1988; Arsac, 1989; Laborde et Capponi, 1994; Balacheff, 1999. In the case of geometry, the concrete object is often a diagram. To understand the differences between the student and the teacher in its exploration, these authors often consider two different objects: a concrete object, *the drawing*, which is a material representation of a specific formal object, called *the figure*, which corresponds to the class of drawings representing the same set of specifications. Not only DG environments can have a role in helping learners to move from *the drawing* to *the figure* but also they may be used as a tool to help students to conjecture and to understand the necessity of proof.

On the other hand, the idea that DG's limitations may be used as a powerful tool for the development of deductive reasoning in geometry was consistent with the notion of conflict discussed in the previous section. This conclusion led us to look for theoretical similarities between our previous studies on theoretical-computational conflicts and on Dynamic Geometry environments. On that account, we present in this article a case study on a student's reaction to a DG environment and search for understanding the results on the light of the notions of *description* and *conflict*.

THE PROBLEM

In this section, we briefly present the problem used as main the task in the case study to be analyzed. This problem has been applied many times as an activity in in-service teachers courses in our University, when samples of participants' reactions have been tape-recorded and a consistent behavior has been recognized. We summarize below the typical behavior of in-service teachers when dealing with this task.

The search for activities that can move our teachers away from meaningless application of formulae and towards conceptual understanding led us to explore historical examples of concept development. One of such examples is the problem of "parallelogrammic areas", which can be found in *The Elements* of Euclid, book I (Heath, 1956, vol. I, p.325). Euclid's study of areas in book 1 start from proposition 35, as shown in figure 1, and ends by the proof of Pythagorean Theorem and its converse (propositions 47 and 48).

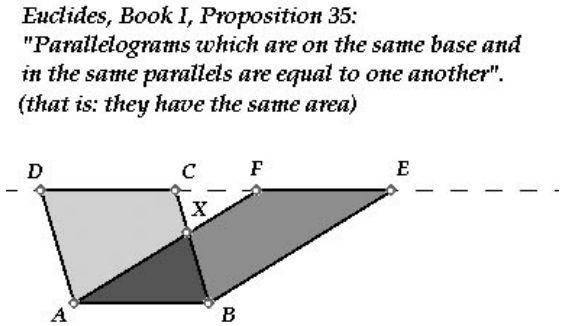

Figure 1: Euclid's Proposition 35, book 1, in DG environment

Instead of assigning numeric values for area measurement, the arguments used by Euclid to prove every result in this sequence is based on intuitive axioms (such as: congruent figures have equal areas) and logically deducted from previous theorems (including congruence of triangles). In order to explore these results in a computer environment, we prepared a series of sketches using *Tabulæ* software (Barbastefano et al., 2000), which allowed the students to explore a large number of examples, by modifying the parallelograms. As the DG software allowed the students to numerically evaluate the areas, the modifications in the parallelograms could be used as an illustration of the equality of the areas whenever the pair of parallelograms satisfied the conditions of the theorem.

In this episode, participants do not deal with a theoretical-computational conflict from the same nature as the one reported in the maximum area problem. The numerical evaluation of the two areas made by the DG software are equal for each pair of parallelograms satisfying the hypothesis of Proposition 35, confirming the mathematical result. Our aim with this activity was to motivate students to develop arguments to justify the result, without using the well-known formulae. In order to do so, the exploration was supported by written guidelines.

We have applied this activity with different groups of mathematics teachers. As a general pattern, they start to work by using the formulae and they do not seem to be aware of the possibility of constructing a non-numerical solution to the problem. That is: the idea of area is strongly attached to area measurement. Once they overcome this

initial obstacle, in-service teachers seem to profit from the experience and become more aware of area as a concept, independent of its measure. It was also observed that once they start to grasp the validity of Euclid's arguments of proof, they become able to apply these arguments to prove other results of the sequence of propositions. It is also interesting to remark that not a single time any student brought about the discussion that the figures moving on the computer screen suggest that the perimeter of the parallelograms might tend towards infinity whilst the area remains constant.

THE CASE STUDY

The episode we analyze here was observed when we were testing a set of activities to be used in in-service teachers courses. The subject is a last year secondary student, Luiza (pseudonym), with a performance in mathematics considered as below average by teachers. On the other hand, she was considered one of best performance students by her teacher of "ruler and compass constructions" (which, in Brazil, is studied as topics of Graphic Arts in secondary school). In particular, she acknowledged that although she had mathematical lessons on area measurement, she was not familiar with usual formulae – unlike the participants of our teacher training courses. Thus, the results of this case study strongly differ from the typical pattern reported above.

We will analyze Luiza's reactions when dealing with the activity based on the computational description for Euclid's Proposition 35. She was given a screen as the one displayed on figure 1 and the written guidelines. She was then asked to explore the sketch and to decide whether or not the areas of the two parallelograms were equal. Before reading the guidelines, she dragged the vertices of the parallelograms, pulling one of them away to obtain a picture with two very different looking polygons, as displayed on figure 2. She commented:

Luiza These areas can't possibly be equal! One of them is much bigger than the other! [Points to the parallelogram with longer sides on the screen.]

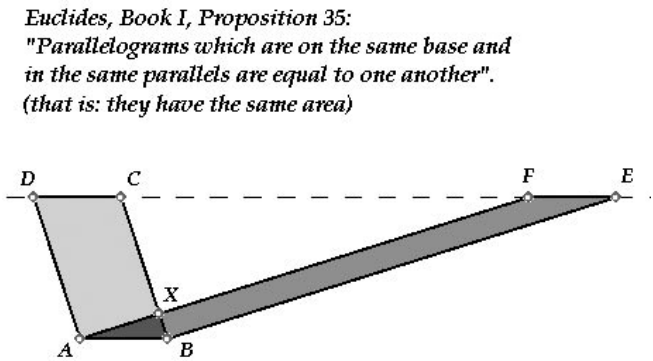

Figure 2: Luiza's interaction with the sketch for Proposition 35.

Luiza started to follow the instructions on the guidelines, which asked the students to verify step by step the arguments of Euclid's proof for proposition 35. She had no trouble to recognize the congruence of triangles ADF and BCE, and she immediately jumped to the conclusion that they had the same area. She had no trouble either to understand that if the area of the triangle XCF was subtracted of the area of each of triangles, and the area of triangle ABX was added to the remains, the results of these operations were the areas of the two parallelograms pictured in the sketch and, therefore, they had the same area.

After she reached the conclusion, the interviewer modified the parallelograms and asked her if the results still held. She felt no need to repeat the arguments step by step and commented that it was obviously true. It is also important to notice that Luiza became completely convinced by Euclid's arguments that the two areas were equal, giving up her previous conceptions, which were merely based on visual observation of the different shapes. In our interpretation, it is not the case to conclude that, by the end of the episode, Luiza gave up her beliefs on her visual observation. Rather, she has aggregated the logical arguments based on visual perceptions used by Euclid to expand her capacity to understand the concept of area. It became clear to her that different shapes can have equal areas.

FINAL REMARKS

Throughout our experience with in-service teachers training, a consistent behavior has surprised us: almost without exceptions, participant teachers were unable to conceive the notion of area without the idea of measurement. In the theoretical construction, area is first defined as a geometrical property – an attribute of polygons – which can further be represented by a real number for classes of equivalent (congruent) figures. Such behavior suggests that the description for areas by means of area measurements has been overused, leading to a narrowing effect on teachers' concept images.

Luiza's different background enabled her to experience a *conflict*: the geometrical *description* suggested to her that the shapes could not have equal areas. In the case of the teachers in the in-service courses, such conflict is not even actualized – for them area is merely width times height and this description overshadows any potential space for query. As we have commented, the conflict situation has led Luiza to aggregate logical argument as tools for understanding the concept of area. In particular, Luiza's prompt conclusion that the result she just explored would hold for a new pair of parallelograms suggests that she has shifted her conception of this diagram from drawing to figure (in the sense of Laborde & Capponi, 1994). Furthermore, the conflict has given her means to grasp something meaningless before: that parallelograms with same width and height have equal areas. This effect is suggested by her own comment, in the end of the interview:

Luiza Why didn't anyone teach me like that before?

To conclude, let us revisit the final paragraph of our report on the in-service training teachers' reactions to this activity. We stated that, at the first moment, they do not recognize that we are presenting them to an actual problem. In fact, as these teachers seem to consider the area formulae as "axioms", we are presenting them to no real conflict, but to a simple application of well-known previous results. So, the initial obstacle to overcome when developing this activity with these teachers is to bring to light at least two underlying conflicts: (1) the difference between area as an attribute of bi-dimensional shapes and area measurement (better still: one among several of its possible measurements); and (2) the inherent difficulties on the process of obtaining the formulae up to the final generalization to real numbers.

Acknowledgments

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STUDENTS PROBLEM SOLVING AND JUSTIFICATION

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Carolyn A. Maher — Rutgers University

This paper reports on methods of students' justifications of their solution to a problem in the area of combinatorics. From the analysis of the problem solving of 150 students in a variety of settings from high-school to graduate study, four major forms of reasoning evolved: (1) Justification by Cases, (2) Inductive Argument, (3) Elimination Argument, and (4) Analytic Method (use of formulas.) The predominant method for students was reasoning by cases where they used the heuristic of controlling for variables or a recursive argument. Only graduate students and one senior undergraduate student[1] correctly used analytical methods.

INTRODUCTION

It is hardly disputable that justification and reasoning about solutions is an important goal for students doing mathematics. In recent years, attention to student thinking has suggested that there are rich and creative differences in students' approaches to problem solving and in students' supporting the solutions they pose. The purpose of this study is to describe several of these approaches from a diverse population of students in the area of combinatorics. In particular, we will: (1) analyze the different approaches used by the students to solve the problem and justify their solution; (2) consider how the challenge to justify triggered in students' reflection on their reasoning, and (3) present arguments from a wide variety of students ranging from second year high-school to graduate study.

THEORETICAL FRAMEWORK

In order to reach conclusions about a student's level of understanding a teacher must encourage students to justify what they say and do to reveal their thinking and logic (Pirie & Kieren, 1992). Too often, in traditional mathematics classrooms, the answer key or the teacher is the source of authority about the correctness of answers, and unfortunately, quick, right answers are often valued more than the thinking that leads to the answer. Requests to explain their thinking are posed to students frequently only when errors have been made. Sanchez and Sacristan concluded from studying the written work of students that students are not accustomed to expressing mathematical ideas, and offer as an explanation that the emphasis mainly is on producing correct solutions (2003). One consequence is that students develop the belief that all problems can be solved in a short amount of time and they will not persist if a problem cannot be solved quickly. In a survey by Schoenfeld (1989) of high school

[1] Robert was a senior undergraduate student who had extensive experience working with tower problems as a high-school student in Rutgers University longitudinal Study.

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students who were asked to respond to the question of what is a reasonable amount of time to work on a problem before they knew it was impossible, the largest time response given was twenty minutes and the average was twelve.

Another factor that might explain students' hesitancy and discomfort in justifying ideas is a de-emphasis on explanation of problems that are correctly solved. McCrone, Martin, Dindyal, & Wallace (2002) argue for a change in pedagogy in which teachers focus on the problem structure and the justification. They suggest that in doing so, students will have a better understanding of the underlying mathematical concepts and will develop a better sense of the need for proving.

It is our view that the call for explanation and justification triggers in students the need for sense making and reflection. Problems posed to students that require accountability of their ideas lead to successful justification of them.

THE STUDY

The following problem was originally posed by a tenth-grade student, Ankur, to, four classmates in fall 1997. Ankur and his classmates were working together as part of a after school component of a Rutgers University longitudinal study (Maher, 2002).

> How many towers can you build, four high, selecting from cubes available in three different colors, so that the resulting towers contain at least one cube of each color? List all the possible towers. Justify that you have them all.

When the problem was originally presented by Ankur, the students partitioned themselves into 2 groups and 3 forms of reasoning evolved. Since 1997, the same problem was then given to several cohorts of students enrolled in liberal arts mathematics and in graduate mathematics-education courses. Students presented their written work and were invited to give further verbal explanations and clarifications of their solutions. Researcher notes provided the data for the oral explanations. Analysis of the written and oral work of about 150 students indicates the forms of reasoning and justifications offered. In this report the reasoning of 22 students is described. The students have been selected as representative of the larger collection of data.

RESULTS

The justifications that the students used to show that they had indeed found all possible towers can be placed into four major classifications (1) Justification by Cases, (2) Inductive Argument, (3) Elimination Argument, and (4) Analytic Method (use of formulas.) Representative solutions from the high-school (H), undergraduate (U) and graduate (G) students are presented according to the general arguments provided:

Justifications by Cases

<u>(H)-Romina, Jeff and Brian's Solution.</u> They indicated that the set of all possible towers could be partitioned into six groups. Since every tower would have two of one color, they focused on the placement of the duplicate color, using x's and 0's. They

indicated that for each placement of the first, or duplicate color, there would be two possible combinations for the second and third colors. They also indicated that these combinations would have two opposite arrangements for the 2nd and 3rd colors. They then tripled the 12 possibilities to represent every color, concluding that there should be a total of 36.

(U)-Joanne and Donna's Solution: If there are three colors available to make stacks of four, two blocks will always be the same color. Put like cubes in 1st and 4th position. Then put the like colors in positions 1 and 3. Next put the like colors in positions 1 and 2. Next put the like colors in positions 2 and 3.Next put the like colors in positions 3 and 4. Finally put the like colors in positions 2 and 4. There were 6 towers for each position of the blocks of the same color because there are 3 possibilities for the blocks of the same color and two possibilities for the remaining spots that are not taken by the blocks of the same color. Knowing that there are 6 towers for each color combination and 6 color combinations gives 36 towers.

(U)-Rob and Jessica's Solution: Working with two yellow cubes, fix the top cube as Blue and then moved the Red cube into the second, third and fourth positions for a total of three towers. Fix the Red cube on top and moved the Blue cube into the second, third and fourth positions to create three more towers. Place a Yellow cube on top and placed the second Yellow cube in the second, third and fourth position. Each position of the second Yellow cube will produce two towers because the position of the Red cube and the Blue cube can be reversed. This gives six more towers for a total of twelve towers with two Yellow cubes. Repeat this process for two Red cubes and two Blue cubes to give a total of thirty-six towers. There has to be one color that appears twice, while the other two colors appear once.

(U)-Marie's Solution. If the Blue cube appears twice first fix the position of the Blue cube on the top and move the second Blue cube to all possible positions. There are two towers for each position because the other two colors can be reversed. Fix the first Blue cube in the second position and move the second Blue cube into two possible positions. Again each position will give two towers. Finally place the two Blue cubes in the third and fourth position to give two more towers. This process can be repeated for each of the other colors.

(U)-Bob's 2nd Solution. There has to be one color that appears twice, while the other two colors appear once. If the Blue cube appears twice keep the two Blue cubes together and move to all possible positions. There are two towers for each position because the other two colors can be reversed. Next separate the two Blue cubes by one and move into all possible positions. Again each position will give two towers. Finally place the two Blue cubes in the first and fourth position, separated by two cubes, to give two more towers. This process can be repeated for each of the other colors. [Note: Bob originally used an inductive method to produce his towers (see below) and then later gave a cases argument.]

(U)-April's Solution. Start with Blue on the top. If there is also a Blue in the second position, the third and fourth position must be PW or WP in order to have all three colors in the tower. If the second cube is Purple the other two cubes must have at least one White cube. They can be WW, BW, WB, WP or PW. If the second cube is White the other two cubes must have at least one Purple cube. They can be PP, PB, BP, PW or WP. This gives 12 combinations with Blue on top. There are also 12 combinations with White on top and 12 combinations with Purple on top for a total of 36 towers.

(U)-Bernadette's Solution. Place the Blue cube in the first position of the tower. If there are two blue cubes, the second blue cube can be in the second, third or fourth position. There are two towers for each position because the other two colors can be reversed. If there are two Purple cubes they can be together in the 3rd and 4th position or the 2nd and 3rd position or spit between the 2nd and 4th position. The remaining cube must be White. Similarly if there are two White cubes they can be together in the 2nd and 3rd position or the 3rd and 4th position or spit between the 2nd and 4th position. The remaining cube must be Purple. This gives a total of 12 towers with blue on the bottom. There are also 12 towers with Purple on the bottom and 12 towers with White on the bottom for a total of 36 towers.

(G)-Tim's Solution. Given three colors Red Yellow and Green, towers 4 tall containing at least one cube of every color will yield towers with I Red, I Yellow. 2 Green: 1 Red, 2 Yellow, 1 Green; and 2 Red, 1 Yellow, 1 Green. All these cases will be equal in number. Consider 2 Red and 2 Green. There are 6 towers that are 4 tall with 2 Red and 2 Green. Now exchange a Yellow for one of the Red's in each tower. There are two ways to do this for each tower. Therefore there are $2 \bullet 6 = 12$ towers of 1 Red, 1 Yellow, 2 Green; 1 Red, 2 Yellow, 1 Green; and 2 Red, 1 Yellow, 1 Green, for a total of 36.

(G)-Traci's Solution Find all permutations with A on the bottom then all with B on the bottom then all with C on the bottom. From Traci's diagram and annotations one can see that she started by fixing the first three rows as color ABC. Then row 4 can be any one of the three colors. Keeping the first two rows as A and B the remaining two rows can either be AC or BC because we have already accounted for all towers with C in the third row and we have to use all three colors. Thus we have a total of 5 towers with AB on the bottom. She next fixed the first three rows as ACB. Again row 4 can be any one of the three colors. Keeping the first two rows as A and C the remaining two rows can either be AB or CB because we have already accounted for all towers with B in the third row and we have to use all three colors. Thus we have a total of 5 towers with AC on the bottom. If we fix the bottom as AA the top two blocks can only be BC or CB because we must use all three colors. This gives us a total of 12 towers with A on the bottom. There are also 12 with B on the bottom and 12 with C on the bottom for a total of 36 towers.

Induction Arguments

(U)-Errol's Solution. Errol used an inductive method to produce his towers. He said that you could fix the first level as Red. The second level could then be Red, Yellow or Blue. If the second level were Red than the third and fourth level would have the other two colors Yellow Blue or Blue Yellow. If the second level were Blue then the third and fourth level would contain at least one Yellow. It could be Yellow Yellow, Yellow Red or Red Yellow, Yellow Blue or Blue Yellow. Similarly if the second level were Yellow the third and fourth level could be Blue Blue, Blue Red or Red Blue, Blue Yellow or Yellow Blue. This gives twelve combinations which you multiply by three since the first cube could be any of the three colors.

(U)-Christina's Solution. She started by making towers two high by adding A, B, C to each of the three colors. Start with towers with color A on the top. Add a block of each color to each of these towers. Add a block of each color to resulting three tall towers eliminating tower with 3 of one color because it would be impossible to have three different colors. Eliminate resulting 4-tall towers that don't have all three colors. Do the same thing starting with towers with color B on top. Do the same thing starting with towers with color C on top.

(U)-Bob's 1st Solution. Start with six towers that are three-tall with all three colors. Place a Red Yellow or Blue cube on the bottom of each tower. This will give all towers with two of the same color on the bottom and the other colors in all possible positions. Place a Red Yellow or Blue cube on the top of each of the original six towers eliminating the duplicate that you get from having the same color on the top and bottom of the tower. This gives all towers with two of the same color on the top of the tower. [Note: When Bob did the problem this way he missed the towers with the duplicated color in the middle. He found his missing towers when he changed to a cases approach (Glass, 2001).]

(G)-Frances' Solution. Start with the first block as Red. Then the 2nd could be Red, Yellow, or Blue. If the 2nd is Red the third could only be Yellow or Blue. If the third is Yellow then the 4th must be Blue If the 3rd is Blue then the 4th must be Yellow. If the second is Blue then the 3rd could be Red Yellow or Blue. If Red The 4th could only be Yellow. If Blue the 4th could only be Yellow If Yellow the 4th could be Red Yellow or Blue. If the 2nd is Yellow then the third could be Red Yellow or Blue If Red, the last could only be Blue. If Yellow the last could only be Blue. If Blue the last could be Red Yellow or Blue. The same would happen if the first block were Yellow or Blue .

Elimination Arguments

(U)-Penny's Solution. Penny listed all towers four tall with 3 colors using a tree diagram, and then crossed off all towers that did not meet her criteria. Her argument was a combination of inductive reasoning with elimination.

(U)-Robert's Solution. Start with the number of towers four tall with 3 colors, 3^4. Subtract the 3 towers with exactly one color $3(1^4)$. Subtract the towers four tall with

two colors, with at least one of each color. There are 2^4 four tall towers with two colors, but you need to subtract 2 or $2(1^4)$ towers with one color or the other. There are three combinations of two colors (Red/Blue, Red/Green, Blue/Green) I multiplied it by three. So the number of towers with two of three colors at least one of each color, four tall r is $3(2^4 – 2(1^4)$. The total number with at least I of each color is $3^4 – 3(2^4 – 2 (1^4))$.- $3(1^4) = 3^4 – 45 = 36$.

(G)-Liz's Solution. Start with $3^4 = 81$ towers 4 tall when choosing from 3 colors. Subtract out the ones that don't have at least 3 colors. There are $2^4 = 16$ with just Red and Green, 1 all Red and 1 all Green. There are $2^4 = 16$ with just Red and Blue, 1 all Red and 1 all Blue. There are $2^4 = 16$ with just Green and Blue, 1 all Green and 1 all Blue. There are three duplicates. So there are $3 \bullet 16 – 3 = 45$ without at most 2 colors. $81 – 45 = 36$ with at least 1 of each color.

(G)-Mary's Solution. Consider towers 4-tall choosing from 3 colors. $3^4 = 81$. At least one of each color must be present. Go back to towers 4 high choosing from 2 colors. Red and Blue, $2^4 = 16$. Red and Yellow, $2^4 = 16$. Yellow and Blue, $2^4 = 16$. Since only 2 colors are represented in each of these cases subtract. $81 – 3(16) = 33$. I subtracted too many. Red and Blue, Red and Yellow, Yellow and Blue. Each tower of one color appears twice so add three back in and end up with 36.

Analytic Method

(G)-Leana's Solution. I used a numerical formula. How many ways can you arrange AABC = 4! Divide by 2! To eliminate repeats. You get 4!/2!= 12 towers when A is the color repeated . The same when either B or C is the color repeated for a total of 36 towers.

CONCLUSIONS

The forms of reasoning displayed by the students in this study can be placed into four major categories; however there was a great deal of variation within these categories and there is also some overlap between categories. The majority of students that used an elimination method used formulas to calculate the number of towers. The other student that used an elimination method used an inductive method to generate her list of all 81 towers that were 4 tall with three colors. All but two of the students who choose to do a justification by cases did so by controlling for variables. Marie and Bob, instead, used a recursive argument in which they focused on a fixed cubed and rotated it exhaustively for particular cases. The approach to arguing by cases varied. Students chose different cases into which to separate the towers and different variables for which to control as they built their justification. There were also variations within the other approaches. For example, students started their inductive argument at different tower heights. Errol and Francis started at height one; Christina started with height two; and Bob started at height three, but missed some of the towers as a result. He eventually resolved the discrepancy when he considered the method of cases. Bob was the only student that used two different methods and it served him well in finding his discrepancy from the inductive method.

The sharing of ideas was an important component in students' problem solving. It provided them with the opportunity to review their work, reflect on their ideas, and sometimes to modify their results. While the written work does not show the interchange of ideas that came about as students shared their work with others, the invitation to students to share their ideas resulted in a more careful review of the work and a greater confidence in the reasoning offered. For example, it was only after sharing her justification with the instructor that April became confident that she had indeed found all possible towers. In some cases, the discussion revealed to students flaws in their reasoning, resulting in a re-examination of the solution offered. As an illustration, the process of justifying that he had the correct number of towers enabled Bob to realize that his inductive method of producing towers had caused him to miss several combinations. In this context, we can observe how the process of justifying ones' answers can enable students to reflect upon what they have done and whether their answer is reasonable. Limitations in space prohibit a presentation of the interchange of ideas that came about as students shared their work with others.

While the solutions generally fell into the four categories, the distribution of correct solutions was not uniform according to category. Few students used formulas and most of those students also used an elimination argument. The correct use of formulas was limited to graduate students and one senior under graduate student. Undergraduates successfully used arguments by cases and induction, and the predominate method of solution was reasoning by cases.

DISCUSSION AND IMPLICATIONS

Rich problems can be challenging and engaging for students at a wide range of levels. Ankur's challenge, a problem initially proposed to a group of high-school students, has turned out to be of interest to students at many levels and has resulted in multiple kinds of thoughtful arguments. An important feature of this problem was to account for all of the towers and then to build arguments that are convincing to oneself and others. It may be that problems that call for explanation and justification trigger sense making in students. We suggest that multiple opportunities for students to express, revise, and share in writing, and in a verbal exchange of ideas are important contributors. Therefore we recommend that instructors consider writing problems that invite students to explain and justify their ideas in writing and in the verbal sharing of results.

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DIDACTICAL KNOWLEDGE DEVELOPMENT OF PRE-SERVICE SECONDARY MATHEMATICS TEACHERS

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We present the results of a study on the didactical knowledge development of pre-service secondary mathematics teachers participating in a methods course. In this course, we expected pre-service teachers to learn and use a series of conceptual and methodological tools that could help them in the design of didactical units. We coded and analyzed the information contained in the transparencies used by the teachers while presenting their solution to a series of tasks proposed in the course. Four stages of didactical knowledge development were identified and characterized. The evolution in teachers' performance over time is described based on those stages.

DIDACTICAL ANALYSIS, DIDACTICAL KNOWLEDGE AND DEVELOPMENT

Recent discussion about teachers' knowledge originated on Shulman's (1986, 1987) proposals on pedagogical content knowledge. Several authors, including Shulman, have proposed taxonomies of teachers' knowledge, as an approach to characterize this knowledge (e.g., Bromme, 1994; Morine-Dershimer & Kent, 2001). Simon's (1995) proposal is somehow different, defining teacher's knowledge as the knowledge required to plan and implement lessons. That approach comes from a functional point of view.

We undertake a similar functional approach by focusing on the didactical analysis that the teacher carries out to promote students' learning. Didactical knowledge is the knowledge that the teacher uses and puts in practice (and develops) while performing the didactical analysis (Gómez y Rico, 2002). Didactical knowledge involves a series of conceptual and methodological tools that enable the teacher to examine and describe the complexity and multiple meanings of the subject matter, and to design, implement, and assess teaching/learning activities. In the methods course under study, these tools were organized around four types of analyses: content, cognitive, instruction and performance. This study focuses on the knowledge necessary for performing content analysis.

Content analysis is the analysis of school mathematics, that's say the mathematics viewed from its school teaching and learning perspective. Content analysis tries to understand the complexity of mathematical subject matter by focusing on its different meanings. In the case of the methods course under study, the content analysis proposed takes into account three approaches: conceptual structure, representation systems and phenomenological analysis. The conceptual structure is the description, in terms of concepts, procedures and the relationships among them, of the mathematical structure being analyzed (Hiebert & Lefevre, 1986). We see the representation systems as a means for expressing and highlighting different facets of

the same mathematical structure and we work with them under the assumption that they follow a sequence of rules originating in mathematics, in general, and in the specific mathematical structure, in particular (Rico, 1996). The phenomenological analysis involves the identification of the phenomena that are in the base of the concepts, the situations that can be modeled by the mathematical structure, the substructures of that structure that serve as models for those phenomena and situations, and the relationships between substructures and phenomena (Freudenthal, 1983).

Within the context of this course and in relation to content analysis, we see learning as the process in which pre-service teachers develop the necessary competencies for analyzing and interpreting a mathematical subject in terms of the above-mentioned notions, and for using the results of this analysis in the other phases of the didactical analysis and in the design of a didactical unit. We expect progress in learning to express itself in terms of an increasing complexity, variety and structuring of the multiple meanings with which the mathematical subject can be described with the help of the given notions and in a coherent and justified use of those meanings in the other phases of the didactical analysis.

The study followed the general ideas about cognitive development (e.g., Carpenter, 1980) and conceptual change (e.g., Schnotz, W., Vosniadou, S., & Carretero, M., 1999), by assuming that teachers' didactical knowledge development can be described as a process of change in terms of a sequence of stages. Our interest was descriptive. We hoped that the attributes characterizing those stages, and its use for categorizing teachers' tasks to those stages, would allow us to describe how the pre-service teachers progressed in their learning of the three notions composing the content analysis, of the relationships among those notions, and of the use teachers could make of them when designing didactical units. For instance, an attribute characterizing those stages could be the number of representation systems appearing in each of the tasks carried out by the teachers. A small number of them could be a distinctive feature of an initial stage in teachers' didactical knowledge development. An increase in that number might be a feature of posterior stages of development. Based on the conceptual framework of the study and our experience as teachers' trainers, we identified a list of attributes of the work produced by the pre-service teachers during the course. These attributes were organized in terms of different levels of complexity and structuring of the conceptual structure, of variety of representation systems, connections, phenomena and models, and of use of that information in the other tasks of the course.

PROBLEM DESCRITION

We can now establish our research problem as follows: to identify and characterize a sequence of stages of pre-service teachers' didactical knowledge development and describe how the changes in teachers' performance can be represented in terms of

those stages. In what follows, we describe the instruments we used for collecting, coding and analyzing the data.

The study was done with last year mathematics students in a methods course. During the second half of the course, pre-service teachers were organized in eight groups of 4 to 6 individuals. Each group chose a mathematical subject (e.g., quadratic function, sphere) and worked on that subject following the didactical analysis procedure. This procedure involved nine tasks over a five months period, including the final project in which each group proposed a didactical unit design. Each task was presented by each group to the rest of the class with the help of overhead transparencies. Our basic unit of analysis was the information contained in those transparencies. Each transparency presents schematic information about the analysis done by the group of its mathematical subject using one of the notions involved in didactical analysis (phenomenology, representation systems, materials and resources, etc.).

From the list of attributes described above and an exploratory analysis of the collected information, we defined a set of coding variables. These variables established the existence (and in some cases the number of occurrences) of an attribute in a transparency. The following are some examples of those coding variables: the numeric representation system appears in the transparency, number of connections among representation systems, the representation systems organize the conceptual structure, etc. We produced a list of 121 variables, which were used to codify the 72 sets of transparencies.

We wanted to summarize the information that resulted from this coding in order to: (a) identify and characterize a small number of didactical knowledge stages; (b) recognize the degree to which the information in each transparency matched the characteristics of the stage it was assigned to; and (c) determine whether, for a given group of pre-service teachers transparencies, the information contained in them indicated an evolution in time. In other words, we wanted to identify and characterize a group of attributes defining a sequence of development stages that could allow us to explore progress in learning as described above. These attributes had to come from a reduced number of variables originating on the coding variables.

Taking into account the conceptual framework of the study, our experience as pre-service teachers' trainers and the results of the information coding, we produced a set of 12 variables for summarizing that information: 1) number of levels of the conceptual map describing the subject; 2) existence of central notions in the conceptual structure; 3) number of organizational criteria of the conceptual structure; 4) coherent use of the organizational criteria; 5) number of connections in the conceptual structure; 6) number of representation systems; 7) role played by the representation systems as organizers of the conceptual structure; 8) number of phenomena mentioned; 9) number of contexts to which those phenomena belong; 10) number of substructures used to organize those phenomena; 11) role played by the notions of the content analysis (conceptual structure, representation systems and phenomenology) on the other analysis (cognitive, instructional and performance) and

the design of the didactical unit; and 12) coherence between what is proposed in the conceptual structure and the use that is made of it in the rest of the didactical analysis.

Given that, from the perspective of this study, the course was structured in four phases (one phase for each notion of the content analysis and a fourth phase in which these notions were used together on the rest of the didactical analysis), we decided to start the cyclic procedure that we describe below with four stages (note that this procedure shares many similarities to K-means clustering).

The values of each variable are divided in ranges. An observation is an n-tuple of values (x_1, x_2, …, x_{12}), where x_i is the value of the variable i (e.g., number of phenomena) assigned to the information contained in the corresponding transparency. These values were obtained from the original coding of the information. We wanted to identify and characterize the development stages in terms of these 12 variables in such a way that the sequence of stages represented the evolution of the observations and produced a grouping of those observations. A stage S_j was an n-tuple of value ranges of the variables (r^1_j, r^2_j, …, r^{12}_j), where r^i_j was the values range for the variable i (e.g., [2,4]: there are 2, 3 o 4 phenomena in the transparency). Thus, the stage j is defined by the set of all the ranges of order j of the variables. Once the stages are initially defined, each observation is assigned to the stage generating the minimum number of discrepancies. When assigning an observation to a stage, a discrepancy in a variable appears if that variable takes values that do not belong to the range established for that stage. Therefore, the problem becomes one of establishing a definition for the stages that minimizes the number of discrepancies with an acceptable degree of discrimination among them.

METHODOLOGY

We devised a cyclic process for this purpose. Each cycle involves two steps: assigning observations to stages and redefining the ranges for some variables and stages. In the first step, each observation is assigned to the stage that generates the minimum number of discrepancies. In the second step, the variables with the greatest number of discrepancies are identified together with the stages in which those discrepancies are generated. Next, the consequences of changing the definition of those stages (and possibly contiguous ones) in terms of those variables are analyzed. The change in ranges follows a double criterion: reducing the number of discrepancies, while maintaining an acceptable level of discrimination among stages. Once this is done, the observations are reassigned to the new stages. This starts a new cycle. The process stops when the changes in the definition of the stages in terms of the variables needed to reduce discrepancies involve an unacceptable loss in the stages' discriminatory power.

The above procedure, that we call discrepancy analysis, generates a definition of stages that adjusts reasonably to the observations and does not require (as cluster analysis does) that the numerical differences of the variables make sense (in our case,

for instance, the difference between 1 and 3 representation systems was not equivalent to the difference between 5 and 7). This is because the procedure allows (and requires) researchers to use their judgment (based on the conceptual framework and their experience as teachers' trainers) when deciding how to change the range of variables in order to reduce discrepancies, without an unacceptable loss in the discriminatory power of the corresponding stages. Discrepancy analysis neither takes into account whether the discrepancies belong to the stage above or below the stage to which the observation is assigned (direction of the discrepancy), nor gives a different weight depending on the magnitude of the discrepancy.

The results of the discrepancy analysis were used to define a new set of variables satisfying the requirements of cluster analysis. We defined 12 new variables in terms of the ranges that characterize the stages. For a given variable, we assigned the value 1 if the value of the discrepancy variable belongs to the first range (stage 1), the value 2 to the second range, and so on. Based on these new variables, we produced a new set of observations. Each observation is now n-tuple of 12 ordinal values between 1 and 4. In fact, stage 1 is defined by an n-tuple whose values are all 1. We used hierarchical cluster analysis with the Ward's distance definition to produce a grouping of these observations in four clusters. Given that this method is very sensible to outliers, we excluded those observations with more than two discrepancies.

RESULTS

Using the assignment of observations to stages resulting from the discrepancy analysis, we can describe the results of the cluster analysis as follows: (1) there is one cluster containing one of the two observations belonging to stage 1; (2) the other observation from stage 1 (having two discrepancies, one of them of magnitude 2), together with three observations of stage 3 (having either two discrepancies or one discrepancy of magnitude 2) are grouped in a second cluster that contains all 19 observations from stage 2 (except one, see below); (3) the third cluster contains 19 observations from stage 3; (4) the fourth cluster contains the 9 observations from stage 4, together with 5 observations from stage 3 and one from stage 2, all of them having discrepancies in the variable "coherence".

These results highlight the fact that discrepancy analysis neither takes into account the direction of the discrepancies, nor their magnitude, whereas cluster analysis does. It also shows the central role played by the variable "coherence" in the definition of stage 4. The results of cluster analysis lead us to maintain the overall structure of the stages generated by the discrepancy analysis. We can now describe the four stages of didactical knowledge development of content analysis resulting from these analyses.

Stage 1 is a basic stage in which the conceptual structure has no complexity, several organizational criteria are used without any coherence, and at most one representation system is used, without any connections. Stage 2 is a transitional stage. It presents a slightly better organized and more complex conceptual structure in which there is more than one representation system and some connections among them. Stage 3 is

represented by a complex conceptual structure organized by a variety of representation systems, with many connections among them. There is some complexity in phenomenology. Stage 4 presents full complexity in phenomenology and shows that the information collected for the three notions is used coherently in the other phases of the didactical analysis.

Table 1 shows the assignment of observations to stages in the discrepancy analysis. The number in a cell is the stage to which a transparency (columns) of a group of teachers (rows) is assigned. We have underlined those observations that were excluded from the cluster analysis.

Group / Transparency	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*
1	2	2	2	3	3	4	3	4	4
2	$\underline{1}$	$\underline{2}$	3	3	3	3	3	3	4
3	1	2	2	3	3	3	3	3	3
4	$\underline{2}$	2	2	2	2	3	3	3	$\underline{3}$
5	$\underline{2}$	2	$\underline{3}$	3	3	3	$\underline{3}$	4	$\underline{4}$
6	2	2	2	3	3	3	2	$\underline{3}$	$\underline{4}$
7	2	2	$\underline{3}$	3	3	3	4	4	4
8	1	2	2	2	$\underline{3}$	3	$\underline{3}$	$\underline{2}$	$\underline{2}$

Table 1. Assignment of observations to stages

We observe that the groups of pre-service teachers progress in their didactical knowledge development of content analysis at different rates. The step from stage 2 to stage 3 is attained at different moments (at task 3 for three groups, up to task 6 for group 4). The productions from two groups stabilize in stage 3. Two of the five groups that attain stage 4 do so only in the last task (the design of the didactical unit). Group 8 has an erratic behavior, which seems to be due to organization problems within the group.

DISCUSSION

The methodological procedure used allowed us to characterize a sequence of stages and to assign a stage to each observation. Given that the stages are defined in terms of ranges of the variables, it is possible to identify those combinations of attributes that appear simultaneously in a given stage. In this sense, the sequence of stages is illustrative of the pre-service teachers' didactical knowledge development process. For instance, we observe that a low complexity of the conceptual structure occurs simultaneously with a reduced number of representation systems. When the complexity of the conceptual structure increases, the number of organizational

criteria decrease, the number of representation systems increase and they play a more important role in the structuring of the conceptual structure.

The assignment of the observations to the stages (Table 1) shows an evolution of the pre-service teachers' didactical knowledge over time. This gradual progress starts from a basic stage probably grounded on previous knowledge and teaching experience. The progress is coherent with the sequence in which the different notions are introduced during the course. However, there is a lag between the introduction of the notion and the moment in which the knowledge of that notion is expressed in the teachers' performance. This lag is probably due to a process of assimilation and accommodation that originates with instruction, and develops with the teachers' efforts in performing the tasks assigned to them. For instance, the notion of representation system does not consolidate at the time in which this notion is introduced in class and teachers are asked to put it into play to analyze their mathematical subject. This is only a first step. The knowledge of this notion is consolidated when later tasks involve teachers in putting into play these notions in order to solve other problems (for instance, performing the phenomenological analysis, or designing an assessment activity).

The differences in progress rates among the groups might have different causes. In the case of the step from stage 2 to stage 3, these differences might highlight a difficulty in developing and putting into play the notions of representation systems and phenomenology. Nevertheless, all groups overcome this difficulty. The step from stage 3 to stage 4 is more complex. There are groups that do not attain stage 4, and others that do so only in the last task. This might highlight a difficulty in putting into play the information collected in content analysis while performing the other phases of the didactical analysis and the design of the didactical unit. Neither the instruction, nor the activities proposed to the pre-service teachers enabled all groups to overcome this difficulty.

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LEGITIMIZATION OF THE GRAPHIC REGISTER IN PROBLEM SOLVING AT THE UNDERGRADUATE LEVEL. THE CASE OF THE IMPROPER INTEGRAL

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In this work we show some activities designed with a First Year group of the Mathematics Degree to give the graphic register back its mathematical status and promote its use on the part of the students. In particular, we have chosen the topic related to improper integration to reinforce the use of this register by the students.

INTRODUCTION AND BACKGROUND

Some research has stated the reticence Mathematics students show to use the graphic register when they have to solve problems or explain what they do. In particular, this reticence appears to be bigger at University level. On the one hand, the lack of practice in lower levels make it difficult for them to use this register in a natural way; on the other hand, in Higher Teaching this register is usually accused of being "not very mathematical". However, its use may help to avoid numerous calculi or even may be used as a "control" and "prediction" register for purely algebraic work. Eisenberg & Dreyfus (1991) enumerate three reasons why visual aspects are rejected:

- Cognitive: visual thinking requires higher cognitive demands than the ones needed to think algorithmically.
- Sociological: visual aspects are harder to teach.
- Beliefs about the nature of Mathematics: visual aspects are not mathematical.

Mundy (1987) points out that students usually only have a mechanic comprehension of basic concepts of Calculus because they have not reached a visual comprehension of the underlying basic notions; in particular, he states that students do not have a visual comprehension of the integrals of positive functions being thought in terms of areas under a curve (which confirms Orton's (1983) and Hitt's (2003) outcomes on the dominance of a merely algebraic thought in students, even in teachers, when solving questions related to integration).

Other authors' works (Swan, 1988; Vinner, 1989) reinforce the hypothesis that students have a strong tendency to think algebraically more than visually, even when pushed to a visual thought. These authors consider that many of the difficulties in Calculus may be avoided if students were taught to interiorise the visual connotations of the concepts of Calculus.

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Among our results (González-Martín & Camacho, 2003), in accordance with the previous ones, we observe that, in general, students prefer algorithmic-type headings with clear instructions of what is demanded. Moreover, when the non-algorithmic questions use the graphic register, their resolution produces big difficulties in students (who do not use it regularly) or a high rate of no answers. Many students not even recognise the graphic register as a register for mathematical work. On the other hand, lack of coordination between registers produces difficulties to students and some paradoxes make them hesitate. This lack of coordination, or lack of an adequate use of one of the registers, takes them away both from anticipation tools of results to be obtained and control tools of obtained results. We also detect some difficulties and obstacles (González-Martín, 2002), some of them specific to the improper integral concept, as the *bond to compacity* (inability to conceive a volume, or an area, as finite unless the figure is closed and bounded) and the *homogenisation of dimensions* (a volume is attributed with the properties of the area that generates it by revolution, so it is thought that an infinite area will originate an infinite volume). They both may be aggravated by a lack of coordination between registers.

THEORETICAL FRAMEWORK

Aiming to design our teaching sequence we carried out a cognitive analysis of the concepts at issue, trying to detect some difficulties, obstacles and errors that a traditional teaching generates in the students and a competence model was designed in order to assess several students' comprehension (Camacho & González-Martín, 2002). For this analysis we took into account, essentially, Duval's (1993) theory of semiotic representation systems, but we also considered other author's contributions on the role of errors and problem solving in the theory of representation systems (Hitt, 2000).

When it came to design our activities, we gave great importance to the variations of the typical didactic contract and to the construction of an adequate *environment*[1] for each situation (Brousseau, 1988), so that it produces contradictions, difficulties or imbalances. This initial condition of "no control" should produce an adaptation by the students to try to solve the problematic situation given. To promote this interaction, the environment has been designed in such a way that the student can use the knowledge he has to try to control it.

On the other hand, it has also been designed in such a way that allows the student's work to be as autonomous as possible and his acceptance of the given responsibility. This didactic contract is completely new for our students, so we begin with situations close to them to provoke a gradual acceptance of this new contract. The

[1] We have chosen the term *environment* to translate the French *milieu*.

environments designed allow not only the building of knowledge, but also of mathematical knowledge capable of institutionalisation.

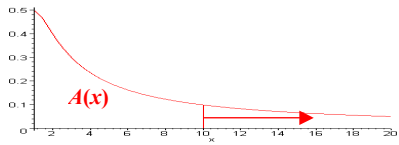
METHODOLOGY

The second stage of our research consists on the design of a teaching sequence that helps to mitigate the numerous lacks observed in our students. However, this sequence also carries out a function of research methodology, which lets us observe and analyse the learning achieved by the students and assess it in an objective way.

The sequence was developed with First Year students of the Mathematics degree and about 25 students took part regularly. Some of the characteristics of our teaching sequence are the articulation of the graphic register with the algebraic one, the reconstruction of knowledge from previously studied concepts (series and definite integrals), the student is given a bigger responsibility in his learning process, the use of non-routine problems (Monaghan *et al*, 1999) and the systematic construction of examples and counter-examples in the two registers.

The graphic register is first presented to interpret some results and later to predict and apply some divergence criteria. On the other hand, we show the students some constraints of this register, which will make necessary the use of the algebraic register. This way, the use of the graphic register, with its potentialities and feeblenesses, together with the use of the algebraic register will facilitate the coordination between both registers.

The limitation to the study of positive functions, in a first moment, and the graphic interpretation of the calculus of areas may justify the definition by means of limits of the improper integral with unbounded integration interval: $\int_a^{\infty} f(x).dx = \lim_{b\to\infty}\int_a^b f(x).dx$.



The study of the behaviour of these two integrals:

$$a)\int_0^{\infty} e^{-x}dx = 1 \qquad b)\int_1^{\infty} x^{-1/3}dx = \infty$$

gives cause for observing that two functions with a very similar graph (in particular when handmade) may enclose quite different areas. The students may think over the possibility to predict *when the integral will diverge*. Is in this situation that the graphic register, if $f(x)$ is positive, allows us to assure that if from a given value on $\boldsymbol{f(x) \geq k > 0}$**,** *the integral will then be divergent***.** This conclusion, together with the two already calculated examples, lets the students see the potentialities of the graphic register to

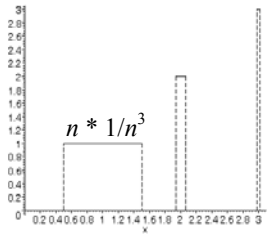
conclude divergence of a given integral and its feebleness to predict convergence, what justifies the development of more formal tools.

The construction of a table with the convergence of the integral of the most usual functions, as the one showed near, lets operationalise both the new definition for the improper integral and the divergence criterion obtained, reinforcing the mathematical status of the graphic register, since the divergence of the integrals indicated with a (*) can be assured using the graphic register.

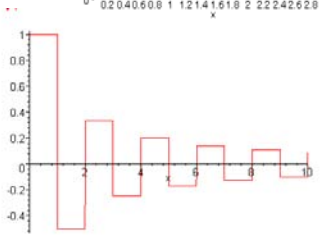
Funtion $f(x)$		**Value of the integral from** $\alpha > 0$ **to infinity** $\int_{\alpha}^{+\infty} f(x)dx$
0		Convergent (= 0) *
a		Divergent *
x^k, $k > 0$		Divergent *
$\frac{1}{x^k}$, $k > 0$	$k \leq 1$	Divergent
	$k > 1$	Convergent
a^{kx}, $a > 1$	$k \geq 0$	Divergent *
	$k < 0$	Convergent
$ln\ kx$, $k > 0$		Divergent *
$sin\ kx$		Divergent
$cos\ kx$		Divergent

Moreover, the fact that this table is constructed between all the students favours their implication in the construction of knowledge and the devolution of the given task, following Brousseau's (1988) ideas. Finally, this table will be used later, when the comparison criteria are studied, so the students will feel participants in the theoretical development of the concepts.

The graphic register and the use of the theory of series also allows the construction of useful counter-examples for questions that usually produce difficulties to the students. For instance, a non-negative function with no limit at infinity whose improper integral is convergent may be built just by constructing over each integer n a rectangle with area $1/n^2$.



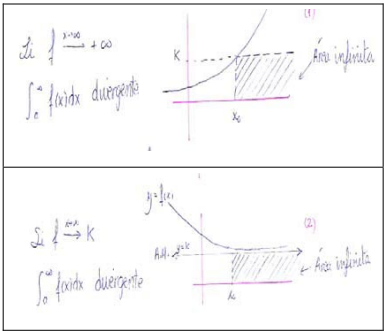
Another quite useful counter-example is provided by the construction of a function whose integral converges, but not absolutely. The classic counter-example is the function $f(x) = \frac{\sin x}{x}$, out of the students' intuitive reach. Using the theory of series and the graphic register, it is much easier to build counter-examples of functions that converge conditionally. In particular, we show a piecewise



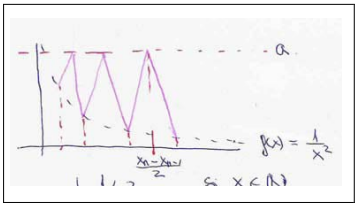
continuous function which over each integer interval [n, n + 1) equals $(-1)^n.\frac{1}{n+1}$. Its integral converge, but the integral of the absolute value coincides with the harmonic series, divergent[2].

DATA COLLECTION, ANALYSIS AND DISCUSSION

Our sequence is assessed in several ways. During its implementation some working sheets are given to the students to be worked out in small groups, answering to new questions using the elements recently introduced; they are also asked to give the teacher a table of convergence of the integrals of the usual functions and the resolution of some problems. The sequence, globally, is evaluated by means of a contents test (with some questions previously used in our preliminary study: González-Martín, 2002; González-Martín & Camacho, 2003). Finally, the students are also given an opinion test about the most relevant aspects of our design.



In our classroom observations we can clearly tell the students' gradual acceptation of the graphic register in order to formulate some conjectures from the moment the *divergence criterion* is illustrated. At the moment of constructing the table of convergences, the students use graphic reasoning to conclude the divergence of the corresponding integrals and state it helps to avoid long calculi. Later, the work carried out in small groups is shared and the teacher gives his approval, what helps to institutionalise this register as a mathematical register. Afterwards, in the sheets given to the students we can see how they use much graphic reasoning. For instance, to analyse the different behaviours of a positive function in a neighbourhood of the infinity in order to conclude the convergence or divergence of its integral; also, to prove the falsehood of the following statement: "$\sum_{n=1}^{\infty} f(n) < \infty \Rightarrow \int_1^{\infty} f(x)dx < \infty$" a group of three students constructs the counter-example shown next[3].



[2] In these activities, we emphasize the construction of the functions graphically and not the obtaining of their formulae.

[3] They create "triangles" joining the points (n, $1/n^2$), (n + ½, a) and (n + 1, $1/(n + 1)^2$).

Furthermore, the students show their satisfaction with the use of the graphic register in their answers to the opinion test (completed by 24 of the students who took part in our sequence) to question 17: " I think the use of graphics as a part of the mathematical work is":

a)	0- Uninteresting. It is not formal	1- Not very interesting	2- Interesting	3- Very interesting and useful
b)	0- It has confused me	1- Indifferent	2- Helps a bit to understand things	3- Helps a lot to understand things

	PREG17A	PREG17B
N of cases	24	22
Minimum	2.000	2.000
Maximum	3.000	3.000
Median	2.000	3.000
Mean	2.458	2.545
Standard Dev	0.509	0.510

In their answers we can observe that two students did not answer the second part of this question. What is quite positive is that the minimums have been, in both parts, 2. Besides, the median in the second part is 3, so more than a half of the students think that its use "helps a lot to understand things".

The exact distribution of the answers is the one shown next.

17-b \ 17-a	2	3
No ans.	**1**	**1**
2	**7**	**2**
3	**5**	**8**

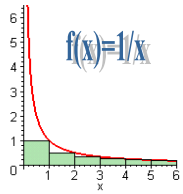
On the other hand, in the contents test, done by 26 students, the questions that needed the graphic register have been answered by a higher percentage than in a group that followed a traditional instruction. For instance, in the second question, which only had *one* correct answer between the 31 participants of the group with a traditional teaching (who interpreted the graph given and sketched a similar one to explain the behaviour of the second integral) we got the following answers:

Question 2:

We know that $\sum_{n=1}^{\infty}\frac{1}{n}=\infty$ and $\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$.

In view of these results, what can you say about the value of $\int_1^{\infty}\frac{1}{x}dx$ y $\int_1^{\infty}\frac{1}{x^2}dx$?

Use the graph given.



- Answers correctly to the first question using clearly the graph: 13
- Sketches a similar graph for the second integral: 6
- Answers correctly to the second question using graphic reasoning: 8

In the third question we got the following answers:

- Answers correctly with graphic reasoning (asymptote): 14
- Answers correctly with algebraic reasoning (criteria): 3
- Infers that the integral of a positive function cannot be negative (area): 2
- Sketches a graph of the function: 3

Question 3:

Can you find any mistake in the resolution of the following integral?

$$\int_2^5 \frac{dx}{(x-4)^2} = \left.\frac{-1}{x-4}\right|_2^5 = \frac{-1}{5-4} - \left(\frac{-1}{2-4}\right) =$$

$$= -1 + \frac{1}{-2} = -1 - \frac{1}{2} = -\frac{3}{2}.$$

CONCLUSIONS

In this work we have shown some activities, related to the topic of improper integration, that try to reinforce the mathematical status of the graphic register in university students. In particular, we believe that the changes in the usual didactic contract (so that the students themselves can see the utilities and limitations of the graphic register) and the work constructing examples and counter-examples, together with the graphic interpretation of results, allow to recognise this register and to accept it. On the other hand, the approval of the teacher reinforces its mathematical status, allowing later institutionalisation.

As said before (Eisenberg & Dreyfus, 1991), unwillingness to the use of this register is quite strong, the cognitive demands it requires are higher. As a consequence, we feel that its use should not be done in an isolated way, but as an habitual part of the instruction, in such a way that the student accepts it and has the "approval" of the teacher. For this reason, its use as a part of an experience (as in our case) is positive, but may just become anecdotic if, once accepted, its use is not reinforced later.

Therefore, some of the open questions that remain are its use regularly during a whole semester, the analysis of the change of the students' attitude towards it and whether they would use it in non-routine questions. Our results, although local, support the hypothesis that undergraduate students may accept it if its utility is motivated and it is used in a reasonable way.

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WILL "THE WAY THEY TEACH" BE "THE WAY THEY HAVE LEARNED"?

Pre-service teachers' beliefs concerning computer embedding in math teaching.

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Embedding computers in math teaching is not a totally new issue, but the dilemmas related to it multiply more rapidly than the answers are being supplied.
One of the dilemmas that we refer to is related to the success of those who are being taught using a certain teaching approach versus their attitude towards the subject they learn. E.g., Funkhouser (2002-2003) found in his research that students who receive geometry instruction using a constructivist approach by means of computer-augmented activities do achieve stronger gains in knowledge of geometry concepts than students who receive more traditional geometry instruction but they do not develop more positive attitude toward mathematics than students who receive a more traditional approach. Actually, for the group that studied math in a traditional way it was one of the preferred subjects, unlike the group that studied it following the constructivist approach with computerized tools. Similar phenomenon was described by Poohkay & Szabo (1995) who studied the achievements and attitudes of pre-service teachers in a primary school program in three groups. One of the groups was taught using animation, the second one was taught using computer, the third one used texts only. The first two groups gave the same grade to the form of instruction they obtained which was lower than the grade the participants of the third group gave, though the achievements of the third group were the lowest of the three.
Norton, McRobbie & Cooper (2000) say that in spite of availability of technology the secondary school teachers rarely used computers in their teaching. They investigated the reasons for this phenomenon. The results indicate that individual teachers' resistance was related to their beliefs concerning math teaching and learning and the existing pedagogies, including their views on examinations, concerns about time constraints, and preferences for particular text resources. It was also found that teachers with transmission/absorption images of teaching and learning and teacher-centered, content-focused pedagogy had a restricted image of the potential of computers in mathematics teaching and learning. By contrast, one teacher with images of teaching consistent with social constructivist learning theory and a learner-focused pedagogy had a broader image of the potential of computers in mathematics teaching. Further, staff discourse was also found to be important in determining whether computers would be used by students to facilitate their conceptualization of mathematics. These findings have implications for professional development related to the integrated use of computers in mathematics teaching.

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Hazzan (2002-2003) related to the attitudes of prospective high school mathematics teachers toward integrating computers into their future classroom teaching. She found that many of the prospective teachers have added a remark in the following spirit "It is worth integrating learning with computers together with learning and teaching without computers". Following Hazzan, a declaration like this indicates that the teachers-to-be don't expect that the computers will resolve all the problems related to the math teaching and learning. They rather tend to consider seriously combining computer in math teaching. Moreover, the students who have already been exposed to the school reality, point at the fact that more experienced teachers hardly encourage their younger colleagues to introduce teaching novelties.

We shall base our reference to the situation on findings both of previous research and on our own one. Our research population consists of pre-service and in-service teachers of primary and secondary school programs. We would like to point out that the link between purposes of a math teaching program and courses constructed in view of these purposes are not always reflected in students' views on these links, see e.g Patronis (1999).

In our opinion, the role of computerized tools in math learning and in developing the math insight of the students is related to numerous factors:

- The rapidly developing computerized environments demand appropriate changes in didactic approaches to be adopted by the teachers or even to be developed by them anew "in real time".
- Variety of math assignments implies a teacher's ability to fit properly a tool to an assignment. E.g., there are several geometric tools of different levels of complexity, those more complex demand more skills to operate them effectively, and the less complex are also less efficient. There are also several computerized algebraic environments etc., some specific topic-oriented computerized tools etc.
- The teacher's professional knowledge must include mastering the variety of computerized tools, mathematical knowledge that would render him flexible enough in his attitudes and responses to the outcomes of the student's activities, and versatility in combining math tools with computerized tools in and optimal way, the optimality itself being a versatile concept.

Keeping all these in mind, we have been asking ourselves for some time: "What is the most appropriate way to prepare instrumentally and mentally the future math teachers to the reality demanding permanent competent adjustment to rapidly developing computerized environment in math teaching?"

In order to try to refer to this question at least partially, we designed a research project in which we studied the performances of several groups of students, studying several courses at different levels of mathematical knowledge and embedding a variety of computerized tools. Moreover, we induced all of the students to experience at least two-three different computerized tools in different courses during three years of their main education program. The embedding of computerized tools occurred in courses in mathematical subject matter courses and in courses in didactics and

pedagogy of math teaching. In addition to these, the students took advanced courses in embedding computers in math teaching.

Our research questions were:

- How effective are our computer-equipped courses in providing the future math teachers with skills and professional qualifications in appropriate computer usage at least for the beginning of their professional career?
- How our students assess their mastering skills in computerized tools developed as a result of theses courses?
- To what extent has their attitude towards computer embedding changed during their years in the college?
- To what extent do they expect themselves to use computers in their future teaching practice?

The students who participated in the experiment, belong to several categories:

- 31 freshmen of pre-service primary school and secondary school programs.
- 27 sophomores of pre-service primary school and secondary school programs
- 12 freshmen and sophomores of in-service secondary school program, who have had previous experience with computer usage in teaching (not necessarily mathematics).

Students of different categories usually studied together in all the courses, thus we could compare their performances and the impact of the courses on their professional beliefs, and follow the changes that these beliefs underwent as a result of the activities.

Materials and methods:

We used three computerized environments in geometry:

- The Geometric Supposer, The Geometer's Sketchpad, The Word drawing tool;

And three computerized tools for algebraic- analytic courses:

- MATLAB, No-Limits, MathematiX.

The students were presented with assignment sheets which included questions of two types: what we regarded "routine" problems and what we regarded "non-routine" problems. The teaching settings were also of two types: separate computer-usage courses in which the students were being trained to use specific mathematical programs, and math courses in which the computerized tools were embedded accounting for the context of the lesson.

The students got an assignment sheet for about 40-45 minutes without an access to a computer; after that, they were encouraged to use the computerized tools familiar to them to try to solve the problems they had not succeeded to solve, and to substantiate the solutions they had found.

The students were asked to answer whether the computerized tools were used for a better understanding of the problems or/and in order to find a solution.

Referring to the routine problems vs. non-routine ones, we decided not to confine ourselves to open-ended problems as a non-routine type in the spirit e.g. of Takahashi

(2000), though his findings seem to be rather conclusive. Keeping in mind the future vocation of our students, we intended to equip them with approaches that would serve both themselves in coping with problems or mathematical concepts they had never come across, and their future teaching activities, providing them with a precious experience of evolving a concept from the very first steps of acquaintance with it.

Hence, we decided to regard as a non-routine any problem that is not familiar to a student, never mind how routine it may be after future sufficient teaching and exercising. The routine problem is accordingly a problem in a familiar topic and the one the solution to which the student can construct on his own, needing no help either from an instructor or from the computer. In this classification of problem, we account for an important aspect of the Van Hiele theory, which is the development of the insight in the students see e.g. Hoffer (1983). Following Van Hiele, Hoffer defines insight as a merge of three main abilities: a) to perform in a possibly unfamiliar situation; b) to perform competently (correctly and adequately) the acts required by the situation; c) to perform intentionally (deliberately and consciously) a method that resolves the situation. Applying newly learned computerized tools both to routine and to non-routine problems creates a situation in which the mathematical insight is invoked and developed, even if the mathematical problem is originally familiar (routine).

In addition to the tests, the students were asked several questions. As we have mentioned earlier one of these questions was: "Do you think you will use these or other computerized tools in your future teaching activities?"

Results and observations

In attempt to examine the effectiveness of our computer-equipped courses in the future professional activity of our math students we first studied the way they used the provided computerized environment in a variety of courses and assignments see Gurevich et al (2003).

The students' responses were analyzed and classified according to the group category and the problem kind. Here we refer to the students' answers only concerning the solution of the non-routine problems. The results show that in the groups that studied various mathematical courses combined with the intensive computer usage in about 69% of cases the students answered that they have used the computerized tools for the better understanding of the problem and in about 93% of the cases they used computer in order to find the solution. On the other hand in one group where the students were only briefly acquainted with the appropriat computerized tools only about 12% of the participants admitted that they used the computers for the better understanding and only in 3% of the cases the computer was used for the solution finding.

We present the selection of typical answers reflecting the common atmosphere and the opinions of the majority of students:

"I had only basic knowledge in computers before the college. Now I can use No-Limits, MathematiX, The Geometer's Sketchpad. I shall use the computers in teaching if it will be technically possible."

"My proficiency in computers did not change, since I was a practical engineer in computers before I came to the college. My attitude towards computers usage in teaching math also did not change, since I have always enjoyed it, and I shall use them if it is possible."

"Before the college, I was acquainted only with basic computer tools. Now I learned to work with No-Limits, MathematiX, The Geometer's Sketchpad, MATLAB, The Geometric Supposer. I master best The Geometer's Sketchpad, MATLAB, and No-Limits. The computer may be very useful for "visual" learners. For example, we have talked and learned about functions, but I actually understood the concept of a limit at a point only when I studied it with the computer. I shall combine computer and the chalk-and-blackboard methods in my future teaching."

" In spite of my positive experience with computers (*for the reasons similar to those of the previous student),* I am not sure I shall use it in my teaching, since too much technical problems are involved: there are no enough computers for all the pupils, no spare equipped rooms etc."

" The computer shows things that it is difficult to imagine: e.g., logarithmic function, or the sum of angles in a triangle that does not change."

" The computer opened new world for me. But still, when it comes to teaching, I doubt if I shall use it. It is too messy to use with the pupils".

" I am good at computers, especially in MATLAB, No-Limits, MathematiX. These packages demonstrate beautifully the things we have learned, e.g. graphs of functions, geometric constructions etc. But should understand that it does not prove things but rather presents them in an unexpected aspect and thus sometimes facilitates the search for the proof. I do think that one should use the computers in presenting the mathematical concepts, but this should fit the system and the class".

Some conclusions

Referring to this selection of answers, we observe several common features:

- We observed a qualitative difference in socio-mathematical aspects of the computer usage between math courses and special courses for computer usage in math teaching. The lessons in the first setting were lessons in mathematics, they were centered about mathematical issues and concepts, and the students regarded computers to be another tool, in addition to the chalk-and-blackboard, sometimes a very useful one. The lessons in the second setting were regarded as lessons in computers, and the mathematics seemed to be of minor importance. Thus, the students learned to work with computerized tools, but remembered very little of what it was all about. This led us to the conclusion that all the computerized tools are to be learned and taught in context – only in such a way this is the meaningful way of study.

- As a result of the previous conclusion concerning the need to merge the two forms of courses, the role of the teacher in such a course will also have to change: The teacher must become a mediator between two types of knowledge: disciplinary knowledge in mathematics, and mastering skills of ever developing media. The more the computerized tools develop, the more they may tempt one to replace the rigorous mathematical concept or reasoning by a more or less precise and very pretty picture. Hence the teacher is to be able to lead a enlightened mathematical discussion delimiting and defining the abilities of the tool.
- The teacher-to-be who has undergone the process of coping with non-familiar mathematical situation being aware of the limitations of computer usage in mathematical learning, is apt to search and find appropriate ways of embedding the computer in his future teaching practice.
- The students appreciate the visual contribution to their learning process. In this aspect, it seems appropriate to relate to vast research literature on the concept of pre-formal proof and pre-formal approach in general to the teaching of mathematics for the students who are at the visual level or are at the transition stage from the visual towards more formal levels see e.g. Blum & Kirsch (1991), Straesser (2001), De Villiers (1996), Pinto & Tall (2002). It is also known from the literature (see e.g. Senk (1989), Mayberry (1983)), that an essential part of freshmen in teaching education programs is at the visual level of perception of e.g. of geometrical notions.
- The students claimed that mastering several mathematical packages was essential in their success and thus they supported the embedding of computers in their own learning process. We assert that among the students who participated in the experiment, those who were at lower levels of mathematical (in particular, geometrical) thinking[1], developed some pre-formal reasoning and proof skills, similar to those they may expect to come across in their future pupils. This has led them to rather positive attitude towards the role of computer in teaching/learning procedures. Among the students who appeared to be at upper levels, we discovered in several of them a gap at the visual geometrical level. These students did not use the computer for their own visualization purposes, but appreciated its potential contribution to the learning of their future pupils who are supposed to use it this way.
- The technical problems related to the computer usage are not negligible. So much so that they may persuade not to apply computer at the classroom even those students who would otherwise be quite enthusiastic about the idea.
- It is important that the lessons in didactics of math teaching include discussions on socio-mathematical norms accounting for contemporary research e.g.Yackel, 2001, Doerr & Zangor, 2000, Goldenberg, 1999. The students who are also future teachers experience the approach to computer embedding that does not contradict the

[1] We presented all the participants of the experiment group with the Van Hiele tests to appraise their level of geometrical thinking, in particular, in order to relate it to their usage of geometrical packages.

traditional mathematics see e.g. Yerushalmy 1991, Galindo, 1998, Hanna, 1998, but rather enhance some of inductive options on the way to a solution or to a proof.

- Another important aspect is the adjustment of computerized tools used in the math course to the level of the students' mastering the computer. As some of the our students' responds indicate, an inappropriate choice of a tool may obscure the idea of the lesson which is sometimes very elementary and accessible for a relatively simple tool, like Excel or Word drawing tool. On the other hand, the students must become aware of the vast range of opportunities that more sophisticated tools bring with them, and be able to utilize these opportunities to the maximal extent. Hence, the teacher who is to lead the calls to use the computerized environment is to master a wide range of tools. This equally refers to the teachers in the college and to the teachers-to-be who are students at present.

The absolute majority of students tend to use the computers in their future practice if the technical side enables that. One of the students pointed out that she will be cautious not to appear too innovative, in fear of not being accepted in the math team of the school. Others feared that they might loose the control of the class and prefer to use the computer for teacher-provided demonstration alone. No one emphasized the teacher's high abilities needed for this kind of teaching. This may indicate at the lacking self-image of the students as future teachers and hence their inability to place themselves where they are to be in a more or less near future.

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PROGRESSIVE DISCOURSE IN MATHEMATICS CLASSES – THE TASK OF THE TEACHER

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This paper uses data from two mathematics lessons to explore the nature of progressive discourse and examine critical features of teacher actions that contribute to mathematics classrooms functioning as communities of inquiry. Features found to promote progressive discourse include a focus on the conceptual elements of the curriculum and the use of complex, challenging tasks that problematised the curriculum; the orchestration of student reporting to allow all students to contribute to progress towards the community's solution to the problem; and a focus on seeking, recognizing, and drawing attention to mathematical reasoning and justification, and using this as a basis for learning.

INTRODUCTION

> Classroom discourse can be progressive in the same sense as science as a whole is progressive. Scientific progress is not one homogeneous flow; it contains innumerable local discourses that are progressive by the standard of the people participating but that, with respect to overall progress in science, may only be catching up or even may be heading in the wrong direction. The important thing is that the local discourses be progressive in the sense that understandings are being generated that are new to the local participants and that the participants recognize as superior to their previous understandings. (Bereiter, 1994, p. 9)

Our interest in classroom discourse arises, in part, from previous collaborative work based on the notion of *Communities of Inquiry*, which underpins the *Philosophy for Children* movement (see, for example, Splitter & Sharp, 1995). Key features of classrooms functioning as communities of *philosophical* inquiry are the development of skills and dispositions associated with good thinking, reasoning and dialogue; the use of subject matter which is conceptually complex and intriguing, but accessible; and a classroom environment characterized by a sense of common purpose, mutual trust and risk-taking. Our concern has been how these features can be made a part of everyday classroom practice in mathematics.

In earlier work, we have reported a high level of support among principals, teachers and mathematics educators for mathematics classrooms functioning as communities of inquiry, together with a realization that current Australian practice falls far short of this goal, partly because the cognitive demands of typical lessons are low and do not challenge children (Groves, Doig & Splitter, 2000; Doig, Groves & Splitter, 2001); and the critical role of conceptually focused, robust tasks that can be used to support the development of sophisticated mathematical thinking (Groves & Doig, 2002). In this paper, we focus on aspects of classroom discourse associated with classrooms functioning as communities of *mathematical* inquiry.

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According to Bereiter (1994), classroom discourse can be progressive in the same sense as science, with the generation of new understandings requiring a commitment from the participants to working towards a common understanding, based on a growing collection of propositions that can or have been tested. In a similar vein, Cobb, Wood and Yackel (1991) contrast discussion in traditional mathematics classrooms, where the teacher decides what is sense and what is nonsense, with genuine dialogue, where participants assume that what the other says makes sense, but expect results to be supported by explanation and justification. Mercer (1995) proposed three forms of talk that can be used to aid the analysis of classroom talk and thinking: *disputational talk,* featuring disagreement and individualized decision making, with few attempts at synthesis; *cumulative talk*, in which speakers build positively, but uncritically, on previous speakers' utterances; and *exploratory talk,* where critical, but constructive, use is made of another's ideas, challenges are justified, and alternative explanations offered. It is this last category of *exploratory talk* that resonates with good thinking, reasoning and dialogue in *Communities of Inquiry.*

This paper uses data from two, apparently quite different, mathematics lessons to explore the nature of progressive discourse and examine critical features of teacher actions that contribute to mathematics classrooms functioning as communities of inquiry.

A YEAR 1 LESSON ON ADDITION IN JAPAN

This lesson, observed by both authors late last year in Japan, was taught by an "expert teacher", Hiroshi Nakano, to a Year 1 class of 40 children. The lesson was part of a sequence of lessons on addition.

The lesson commenced with children being presented with a series of flashcards with shaded and unshaded dots arranged in two rows of five, and children being asked to show how many more shaded dots were needed to "make 10". This was followed by a similar task where the flashcards showed single numerals instead of dots.

The children were then presented with the problem for the day — finding the answer to 8 + 6 and explaining the reasons for their answers. Children worked individually for 5 minutes, after which the teacher wrote 8 + 6 = 14 on the blackboard and invited particular children to write their solutions on the board.

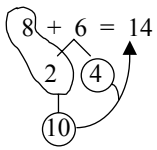

Figure 1: Girl 1's solution for 8 + 6 = 14

Girl 1's solution is shown in Figure 1. When asked, most children stated that they had used the same method. The teacher then asked the children to guess why Girl 1 had

divided the 6 into 2 and 4. Children responded that this was based on "Nishimoto-san's making 10 rule" — apparently formulated by one of the children, Nishimoto-san, in the previous lesson where the problem was to find 9 + 6.

The teacher then asked for a different solution. Boy 1's solution is shown in Figure 2.

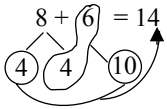

Figure 2: Boy 1's solution for 8 + 6 = 14

The teacher commented that this was again using "Nishimoto-san's making 10 rule", and asked for another way. Girl 2's solution, still described by the teacher as using "Nishimoto-san's making 10 rule", is shown in Figure 3. A few children said they had used this method.

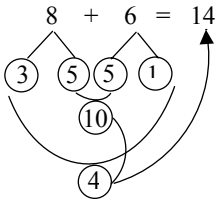

Figure 3: Girl 2's solution for 8 + 6 = 14

Boy 2 stated that he did not use the "making 10 rule". Children tried to guess how he found the answer — had he used a "making 5 rule"? Boy 2 said he had not and explained his reasoning as shown in Figure 4.

8 + 6 = 14

because 9 + 6 = 15 "we did this before"

and 8 is one less than 9 . So, "if 9 becomes 8, the answer is one less".

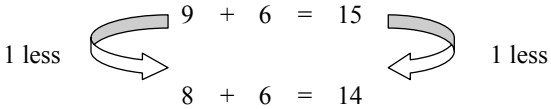

Figure 4: Boy 2's solution for 8 + 6 = 14

Many children clapped in response to this solution and a girl commented that this used their former knowledge of addition.

The teacher suggested that they move on to looking at 7 + 6 using the same method.

Surprisingly, rather than starting with 8 + 6 = 14, Boy 2 again started with 9 + 6 = 15 as shown in Figure 5.

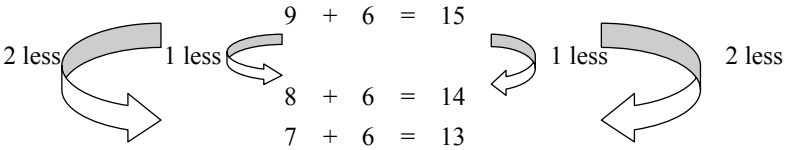

Figure 5: Boy 2's solution for 8 + 6 = 14

The teacher asked everyone to "check the hypothesis" that the answer is 13. Several children demonstrated their solutions using similar methods to those shown in Figures 1 to 3 — i.e. using the "making 10 rule".

Now that children had confirmed that 7 + 6 = 13. the teacher asked them to complete Figure 6, using Boy 2's method and confirming their answers as before.

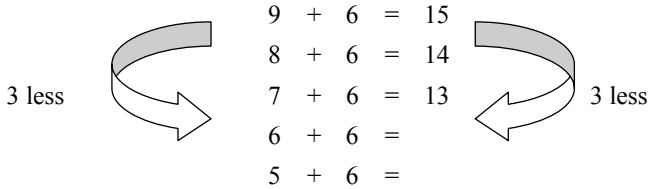

Figure 6: Using Boy 2's solution for 6 + 6 and 5 + 6

One boy continued the list to 0 + 6 and then even further to 10 + 6, 11 + 6, …, 16 + 6.

A YEAR 7 LESSON ON THE AREA OF A TRIANGLE IN AUSTRALIA

This double lesson, taught by Gaye Williams to a class of approximately 24 Year 7 girls in Australia, was videotaped as an "exemplary problem solving lesson" for teaching purposes at Deakin University. The lesson was part of a sequence of lessons on the topic of the area of a triangle. Video extracts will be shown in the presentation to supplement this necessarily brief description of the lesson.

Girls worked in groups of four, trying to find a rule for determining the area of a triangle. One group already knew the rule and was trying to find a rule for the area of a trapezium. The teacher introduced the problem by saying:

> You can draw as many triangles as you like …. What you want to do is to try and find the amount of space inside them; see if you can find any patterns; think about whether those patterns always happen; and try some more if you think you need to try more, until you think you know how to tell someone how to find the amount of space inside a triangle. I mean there might not even be a rule — except these people [the group working on the area of a trapezium] think there is.

The girls were given 10 minutes to make as much progress as they could, before one person from each group was asked to report on what their group was thinking about. Initially, some groups struggled with the difference between area and perimeter and tried to use irrelevant information such as the angle sum of a triangle.

As the groups worked, the teacher moved around the room, asking questions and observing students working, very much in the manner of the Japanese *kikan-shido* "between desk walking" or "purposeful scanning" (see, for example, Kepner, p. 7). As well as using this as an opportunity for selecting the order of reporting, the teacher also sometimes suggested specific aspects she wanted the group to report. While each group could choose who would report, there was an understanding that each member would report at some stage during the investigation.

During the initial reporting, the teacher reminded the girls that they were not allowed to contradict but only to ask for further explanations.

After some considerable time, at least one group came up with the standard rule for the area of a triangle of "base times height divided by two". Commenting on this group's report, the teacher said:

> We have a couple of interesting things here. I had a question to ask, but I didn't need to ask it. I was going to ask "Can they really say they have a pattern when they have only worked with one triangle?" And then Kathryn went on and said they'd worked with heaps of triangles! That's OK. It looks like they really have a pattern. But I hope they looked at some really unusual triangles to make sure it seemed to be happening all the time. But then I *loved* Sarah's question because when you have found a pattern that's the beginning not the end — that's when you have to think "well if it really is so, *why* is it so?"

Before discussing these lessons further, it should be made clear that neither of these teachers is "typical". Nevertheless, the Year 1 lesson shares many features with almost every Japanese lesson observed by us, although the same could certainly not be said about the Australian lesson. Nakano is a well-known teacher whose lessons have been the basis for many Lesson Studies, including one of the video exemplars used in a US-Japan Workshop (see Kepner, 2002; Nakano, 2002), while Williams is the author of a book containing a detailed theoretical and practical approach to learning through investigations (Williams, 1996).

THE TASK OF THE TEACHER

> A good discussion occurs … when the net result … is discerned as marking a definite progress as contrasted with the conditions that existed when the episode began. Perhaps it is a progress in understanding; perhaps it is progress in arriving at some kind of consensus; perhaps it is progress only in the sense of formulating the problem — but in any case, there is a sense of forward movement having taken place. Something has been accomplished; a group product has been achieved.
> (Lipman, Sharp & Oscanyan, 1980, p. 111)

We would argue that in both of these lessons there is progressive classroom discourse in the sense of Bereiter (1994). Moreover, the three key aspects of classrooms functioning as *Communities of Inquiry* could also be observed. We will now discuss what we believe are some critical features common to the two teachers' actions.

Problematising the curriculum

> [Students'] understanding increases significantly with their discovery of concepts they have built out of their own prior mathematical knowledge. (Williams, 1996, p. 2)

In both lessons, the teachers took what is usually regarded, at least in Australia, as a standard piece of mathematics, to be taught by either exposition or what Simon (2003) refers to as *empirical activity*, and transformed it into challenging and problematic, yet accessible content. As stated earlier, the importance of the development of conceptually focused, robust tasks to support the development of sophisticated mathematical thinking should not be underestimated.

In Japan, this is supported through the use of Lesson Study, which aims to research the feasibility or effectiveness of a lesson (see, for example, Kepner, 2002; Nakano, 2002). Moreover, a common framework for lesson planning in Japan uses a four column grid with the first showing the following steps: Posing a problem, Students' problem solving, individually or, less frequently, in small groups; Whole class discussion; and Summing up; possibly followed by Exercise/extension. Each of these is accompanied by entries under the column headings of Main learning activities; Anticipated student responses; and Remarks on teaching (Shimizu, 2002). This common lesson pattern, based on students' actual and anticipated solutions of a single problem, together with an in-depth analysis of these solutions, promotes the problematising of the mathematics curriculum.

As well as *kikan-shido*, referred to earlier, key pedagogical ideas shared by teachers and forming observational criteria include: *hatsumon* — thought-provoking questions important to mathematical development and connections; *neriage* — raising the level of whole class discussion through orchestration and probing of student solutions (Kepner, 2002); and *yamaba* — regarding a lesson as a drama structured around a climax or "yamaba" (Shimizu, 2002).

Establishing an appropriate classroom environment

> Where elegance and originality are valued; the search for the most elegant solution becomes the intrinsic motivation of the group. (Williams, 1996, p. 2)

The classroom environment in both lessons was clearly characterized by a sense of common purpose, mutual trust and risk-taking in the sense of *Communities of Inquiry*. The common purpose was achieved through both the use of a task that was genuinely problematic, yet accessible, for students, and through the establishment of social norms that valued individual (and group) contributions to the solution process.

In the case of the Australian lesson, it was evident that a great deal of effort had been made by the teacher to establish an environment where risk-taking was both supported and simultaneously minimized — for example, as stated earlier, the teacher reminded the girls during a report they were not allowed to contradict but only to ask for further explanations. This was one of many "rules" that formed part of explicit social norms operating in her classroom (see Williams, 1996, for further details). In Japan, while such social norms still need to be established, the fact that there is a

common pattern of lessons and a shared understanding among teachers of key pedagogical ideas, means that students anticipate how a mathematics lesson will operate and do not need explicit instruction on the social norms. Moreover, Japanese teachers frequently make a point of using students' incorrect solutions as a stepping stone to the class developing their understanding. In Australia, a great deal of successful effort has gone into establishing safe classroom environments, although there is very little emphasis on establishing a common (intellectual) purpose, especially when, in many primary schools particularly, groups of students are often working on different tasks — a practice that clearly mitigates against progressive dialogue, at least in the whole-class setting.

Focusing on good thinking and dialogue

> I would like to make my class enjoyable for children's thinking. I want the class to operate so that the children's thinking can be recognized by others and also by teachers. I also like to make the class feel that they can find out about the similarities and differences of their ideas in relation to others. (Nakano, 2002, p. 65)

In both lessons, not only were there well-established social norms relating to discussion, but also, in Yackel and Cobb's (1996) sense, well-established socio-mathematical norms for what counts as acceptable explanations and justifications. Simon (2003) describes a Year 6 lesson also on the topic of the area of triangles as constituting *empirical activity* as opposed to *logico-mathematical activity* and defines mathematical understanding as requiring a "learned anticipation of the logical necessity of a particular pattern or relationship" (p. 185). In contrast to Simon's lesson, the Australian lesson explicitly emphasized the need for this logical necessity when the teacher stated that "when you have found a pattern that's the beginning not the end — that's when you have to think well if it really is so, *why* is it so?"

CONCLUSION

While the two lessons discussed here clearly differ in many respects, there are also many similarities, with the different contexts highlighting the ways in which the teachers promoted progressive discourse. Firstly, both teachers had a clear focus on the conceptual elements of the curriculum and were able to devise and sustain the use of complex, challenging tasks, that problematised the curriculum. Secondly, progressive discourse was promoted through the orchestration of the reporting of student solutions, starting with the least mathematically sophisticated in order to allow all students reporting to progress the community's solution to the problem. This aspect requires the teacher to not only interact with students as they work on the problem, but also to anticipate potential solution strategies and select an order for student reporting. Most of all, progressive discourse was promoted through the teachers' focus on seeking, recognizing, and drawing attention to mathematical reasoning and justification, and using it as a basis for learning. Factors that appeared not to affect progressive discourse in these cases included the age of the students, the mathematical topic, nor the use of co-operative group work.

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CHARACTERIZATION OF STUDENTS' REASONING AND PROOF ABILITIES IN 3-DIMENSIONAL GEOMETRY

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In this paper we report on a research aimed to identify and characterize secondary school students' reasoning and proof abilities when working with 3-dimensional geometric solids. We analyze students' answers to two problems asking them to prove certain properties of prisms. As results of this analysis, we get, on the one side, a characterization of students' answers in terms of Van Hiele levels of reasoning and, on the other side, a classification of these answers in different types of proofs. Results from this research give directions to grade and organize secondary school instruction on 3-dimensional geometry.

INTRODUCTION

The little time devoted by primary and secondary school teacher to teach space geometry is parallel to the little (and quite often wrong) knowledge students show when they have to solve problems in this geometry field. There is an active international research agenda interested in solving several questions related to this problem, like relationships between visualization abilities and the learning of space geometry (Gutierrez, 1996; Kwon et al., 2001; Malara, 1998; Meissner, Pinkernell, 2000), subjects' reasoning processes (Gray, 1999; Guillen, 1996; Lawrie et al., 2002; Meissner, 2001; Owens, 1999), students' knowledge and ways of learning (Jirotkova, Littler, 2002; Lampen, Murray, 2001; Lawrie et al., 2002), improvement of teaching strategies (Lavy, Bershadsky, 2002), benefits for students of using manipulatives (Jirotkova, Littler, 2002) or software (Kwon et al., 2001), problem solving (Lampen, Murray, 2001; Owens, 1996; Stylianou et al., 1999), or theories framing research and curriculum development (Gutierrez, 1996; Owens, 1999; Saads, Davis, 1997).

The research reported in this paper is part of this agenda. Its central focus is to analyze students' level of reasoning and their ability to conjecture and prove in the context of space geometry. As a first step, we have designed a test aimed to provide information about the kinds of outcomes produced by secondary school students when solving problems of proof or conjecture and proof. The result obtained after the administration of the test is a set of students' responses reflecting different Van Hiele levels of reasoning and several types of empirical and deductive proofs.

THE FRAMEWORK AND RELATED LITERATURE

Three elements integrate the theoretical framework of this research: i) The concept

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network of prism is the mathematical content base of the problems solved by students. ii) The Van Hiele levels of reasoning, as characterized in Gutierrez, Jaime (1998) and applying the paradigm of evaluation defined in Gutierrez, Jaime, Fortuny (1991), are used to identify students' reasoning while solving the problems. iii) The categories of mathematical proofs defined in Marrades, Gutierrez (2000) are used to classify the types of proofs produced by students.

i) The two problems we posed to students deal with prisms and some of their elements and properties (see below the statements of the problems). The main concepts and properties necessary to solve the problems are:

* A prism is a polyhedron having two parallel congruent faces (bases) linked by parallelogram faces (side-faces). Prisms can be right prisms, and oblique prisms.
* In any prism, all the side-edges are parallel and congruent. In a right prism, all the side-edges are perpendicular to the bases (and to all the base-edges), and all the side-faces are rectangles. In an oblique prism, no side-edge is perpendicular to the bases, and at least a side-face is not a rectangle.
* A diagonal is a segment joining two non-consecutive vertices of a polyhedron. Diagonals can be face diagonal and space diagonal.
* A n-gonal prism has n+2 faces, 2n vertices, 3n edges, and 2n(n-2) diagonals, n(n-1) of them being face diagonals, and n(n-3) of them being space diagonals.

ii) There is extensive literature describing the characteristics of Van Hiele levels. Some refer to space geometry (Gray, 1999; Guillen, 1996; Gutierrez, 1992; Lawrie et al., 2000, 2002; Owens, 1999; Saads, Davis, 1997). The main characteristics of Van Hiele levels referred to the context of prisms and diagonals are stated below. We used these descriptors to analyze students' answers and identify their levels of reasoning.

Level 1: Students are able to draw some diagonals in a given prism, but they are not exhaustive nor can induce a general relationship. Their explanations or justifications are just a description of what they have drawn.

Level 2: Students induce a formula for the number of diagonals of a n-gonal prism after drawing and counting the diagonals in a few prisms, and they justify it just by summarizing the data they have considered. Students can use the formula they have obtained to calculate the number of diagonals or sides of a given prism.

Students prove that a given conjecture is true (false) by drawing a figure as an example (counter-example) to show that the conjecture is (is not) verified. Their justifications are just a description of what they have drawn. In particular, students use a square (cube) as a counter-example for a rectangle (right prism).

Some times students prove that a given conjecture is true by providing a deductive argument that really proves the converse of the given conjecture.

Level 3: Students induce a formula for the number of diagonals of a n-gonal prism in the same way as those reasoning in level 2, but in this level proofs are abstract

deductive informal arguments (some times based on a specific example drawn) connecting data with the conjecture.

Level 4: Students induce a formula for the number of diagonals of a n-gonal prism by first drawing some specific examples, and then writing a proof. In this level proofs are abstract deductive formal arguments connecting data with the conjecture.

iii) Balacheff (1988) and Harel, Sowder (1998) proposed two well known categorizations of mathematical proofs. More recently, Marrades, Gutierrez (2000) proposed a new set of categories which integrates and expands those defined by the above mentioned authors. We have used the latest categorization to analyze and classify the proof produced by the students participating in our research. Schematically, the categories defined in Marrades, Gutierrez (2000) are:

In empirical proofs examples are the argument of conviction. There are three classes, depending on the *way students select* the examples: Naive empiricism (the conjecture is proved by showing that it is true in one or more, randomly selected, examples), crucial experiment (the conjecture is proved by showing that it is true in a carefully selected, example), and generic example (the proof is based on a specific example, seen as a representative of its class, and it includes explicit abstract justifications.

Each class of empirical proofs has several types corresponding to *ways students use* the selected examples in their proofs: Perceptual proofs are naive proofs involving only visual or tactile perception of examples. Inductive proofs are naive proofs including mathematical elements or relationships. Example-based proofs consist only in showing the existence of an example. Constructive proofs consist in describing the way of getting the example. Analytical proofs consist in using properties empirically observed in the example. Intellectual proofs are based on empirical observation of the example, but the they mainly use abstract properties of the example.

Deductive proofs consist on the use of abstract deductive arguments. There are two classes of deductive proofs, depending on whether students use an example or not: In a thought experiment a specific example is used to help organize the proof. A formal proof is based on mental operations built without the help of examples.

Each class of deductive proofs has two types depending on the *styles of proof* made: Transformative proofs are based on mental operations producing a transformation of the initial problem into another one. Structural proofs consist in sequences of logical deductions derived from the data and axioms, definitions or accepted theorems.

THE EXPERIMENT

To get information on secondary school students' levels of reasoning and proof abilities, we designed an experiment based on the development and administration of a paper and pencil test to evaluate students' behavior and content knowledge in several areas of space geometry. The test has seven items, six of them having the structure of super-item (Collis et al., 1986). The contents of the seven items are: Identification, description, and characterization of solids and their parts (faces, edges,

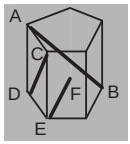
vertices, diagonals); Classification of solids; Cross-sections of solids; Nets of solids; Conjecturing and proving properties of solids.

In this paper we analyze the answers of 299 students from several mixed ability class groups in grades 7 to 11 (aged 12 to 17 years) from three secondary schools in a rural city of New South Wales (Australia). The test was administered to the whole class groups. The Table below summarizes the number of students in each grade and school.

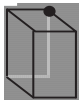
	7th grade	*8th grade*	*9th grade*	*10th grade*	*11th grade*
High School N	12	26	18	35	27
High School O	49	12	23	--	33
High School P	28	--	16	--	20
Total	89	38	57	35	80

We have presented elsewhere the results of the items dealing with nets and cross-sections of solids (Lawrie et al., 2000, 2002). In this paper we analyze the answers to two items asking students to obtain and prove conjectures about prisms:

Item A: a) Remember that a diagonal of a polyhedron is any segment joining two non - neighbouring vertices of the polyhedron. In the figure you can see a polyhedron (a pentagonal prism). Segments AB, CD, and EF are some of its diagonals. Draw three more diagonals of this polyhedron.

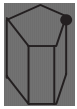


b) How many diagonals has a n-gonal prism (that is, a prism whose base is a n-sided polygon)? Explain, justify or prove your answer.



c) How many diagonals starting from the marked vertex has a rectangular prism? Explain your answer.

d) How many diagonals starting from the marked vertex has a pentagonal prism? Explain your answer.



e) How many diagonals has a n-gonal prism (that is, a prism whose base is a n-sided polygon)? Explain, justify or prove your answer.

f) What prism has exactly 48 diagonals? Explain, justify or prove your answer.

Item B: Tell if the following statement is true or false, and give an explanation, justification, or proof for your answer:

"If all the side-edges of a prism are perpendicular to the base, then all its side-faces are rectangles".

The statement is ..... Explain, justify, or prove your answer.

Question A-a is a reminder of the definition of diagonal of a polyhedron. Question A-b states the main problem without any help. To solve it, students need to reason at

levels 2, 3 or 4, producing different answers depending of their level of reasoning. For those students not able to solve the problem in A-b, questions A-c and A-d are a prompt showing the main clue to elaborate the conjecture and its proof. To answer these questions, only reasoning of levels 1 or 2 is required. Afterwards, question A-e states again the main problem, to check if students are able to solve it after working on the clues. Now, students need to reason at levels 2 or 3. As the answer to question A-e has been guided by questions A-c and A-d, reasoning of level 4 is not required to answer it. Finally, question A-f asks students to apply the result they have obtained in A-b or A-e. Only reasoning of level 2 is required to answer this question.

Item A was presented to students split into two pages, the first one containing A-a and A-b, and the second one containing A-c to A-f. In this way, students do not see the clues while they are trying to solve the problem for the first time (A-b).

Item B asks to prove a given property of right prisms. This is a harder problem, since no prompt is provided to students. Possible answers range from just a drawing followed by a comment to a formal proof, so reasoning of levels 2, 3 or 4 is required.

ANALYSIS OF RESULTS

The objective of the research is to identify kinds of answers produced by secondary school students, so we do not include here quantitative information about frequencies of answers. Below we present examples of the main kinds of answers produced by the secondary school students in the sample.

Answers to question A-b	**V.H. level**	**Type of proof**
An n-gonal prism has 2n diagonals. E.g. a triangular prism has 6 diagonals (2x3 sides). A rectangular prism has 8 diagonals (2x4 sides). [included pictures of a triangular and a rectangular prism and the diagonals of their side-faces]	2	Inductive Naive empiricism
[after counting, with some mistakes, the number of diagonals in several polygons: 4 sides-2 diagonals, 5-5, 6-8, 7-14, 8-19] *From 4-2 and 5-5 obtains d = 3n-10. An n-sided polygon has 3n-10 diagonals. An n-sided prism has two bases (i.e. 2(3n-10)) and 2n diagonals in the rectangles. Then d = 8n - 20 for any prism.*	3	Intellectual Generic example

The first answer exhibits a level 2 reasoning, since a general formula has been induced from some examples. The proof are the examples used to induce the formula, so it is a naive empiricism proof. Furthermore, the student has used the examples drawn to get mathematical information, so the proof is of the inductive type.

The second answer begins with typical level 2 reasoning (inducing the number of diagonals of a polygon) but then it shifts to level 3 reasoning, a generic abstract deductive process to obtain the formula for a n-gonal prism. This proof is built on properties observed in the examples and then stated abstractly, since they refer to a n-

gonal prism, so it is an intellectual generic example proof.

Answers to questions A-c, d, e	**V.H. level**	**Type of proof**
The student drew the diagonals in the given prisms and wrote the numbers of diagonals. No answer to question A-e.	1	- - - -
[In A-c (A-d), the student drew 4 (6) diagonals] *There are 8 (10) vertices; 4 are excluded (the dot and the three others directly beside).* A-e: $d = n - 4$ [without any comment or proof]	2	Perceptual Naive empiricism
[In A-c (A-d), the student drew 4 (6) diagonals and wrote the numbers] A-e: *Each rectangular side has 2 diagonals, so 2n. There are 2 n-gonal bases, so 2n. The space diagonals are n(n-1). Then, d = n(n+3). This doesn't work for prisms where n²4.*	3	Analytical Generic example

Many students produced answers like the first one. They are able to draw and count the diagonals from a vertex, but they are not able to induce a general formula, so they are reasoning at level 1. In this case there is not a mathematical proof.

Students reasoning at level 2 usually solve correctly questions A-c and A-d, as they understand which vertices can/cannot be linked by a diagonal, like second answer. Then they try to induce a general formula to answer question A-e, although they do not provide a reasonable proof of such formula. This behavior is typical of level 2 reasoning, proofs being usually of types naive empiricism or crucial experiment.

The third answer is clearly different from the second one because in question A-e the student wrote a proof of the formula. It is an abstract generic description of the process of getting the formula, based on generalizing the two specific examples of A-c and A-d to a n-gonal prism, so it is an analytical generic example proof.

Students answered question A-f in a meaningful way only when they had obtained a general formula for the number of diagonals of a prism in previous questions. In such case, students used their formula to calculate the number of sides of the given prism, therefore exhibiting a level 2 style of reasoning.

Only a few students in grades 7 to 9 produced meaningful answers to item B, but there were more answers from students in grade 10 and especially grade 11.

Answers to item B	**V.H. level**	**Type of proof**
False: They [the side-faces] *could be squares.*	2	Counter-ex
True: [the student drew a rectangular right prism] *Side edges are perpendicular to the base. Side faces = rectangles.*	2	Ex.-based Crucial experiment
True: The solid could be a rectangular or square prism or a cube. Since squares can be rectangles and the angles in squares and rectangles are 90¡, all side-edges are perpendicular to the base.	2	Converse

False: When you look at more complex solids this statement becomes untrue, like with a dome as structure  *. The side-edges start being perpendicular but then change. This doesn't really agree with the statement. Or a solid as*  *where they are perpendicular but the top of the rectangle is cut off to give jaggard edges. This also doesn't agree with the statement.*	3	Counter-example
True: Prove: That side edges perpendicular make rectangular side faces. Data: Above [the statement of the item]. *Proof: The side edges make right angles with the base edges. The side edges are parallel (all at 90¡ to the base).* *∴ The side faces are parallelograms with 90¡ angles.* *∴ The side faces are rectangles.*	4	Structural Formal proof

Many students, like the first case, considered that squares are not rectangles, so they provided a cube or a right prism with square side-faces as a counter-example for the conjecture. Other students produced more elaborated answers, like the fourth case, drawing prism-like solids and analyzing them to show that the conjecture was false. These proofs cannot be classified into the categories defined in Marrades, Gutierrez (2000) because these categories refer only to proofs of the truth of a conjecture.

Some students made the usual mistake of proving the converse implication, like the one in the third case, showing that they still have not acquired the level 3 reasoning.

Finally, very few students produced formal proofs, like the last case, exhibiting level 4 reasoning. This proof is an example of structural formal proofs, since it does not include any drawing as auxiliary guide to build the deductive argument.

CONCLUSION

An overview of the answers obtained shows a quite complete range of answers in Van Hiele levels 1 to 3. On the contrary, we have only obtained a few answers in level 4, as could be expected from a sample of secondary school students. Research based on university students should be carried out to complete the catalog of answers for the higher levels of reasoning and deductive classes of proofs.

After completing the catalog of answers, the next step in this research program is to design and experiment teaching units focusing on the learning of geometric solids and the improvement of students' reasoning levels and proving abilities.

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TEACHER'S PRACTICES AND DYNAMIC GEOMETRY

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In this paper we analyze instructional materials supported by a Dynamic Geometry software, which were produced by teachers during an in-service training program. We discuss illustrative examples, as well as the outcomes of the critical discussions that took place during the presentations of these materials by the teachers. In order to analyze these materials, our conception of geometry involved a full spectrum of activities, from concrete exploration and experimentation, through conjecturing, problem solving, and on to formal proof. We also took into consideration results from research on the didactical potential of Dynamic Geometry (DG). It is argued that these programs can help us address fundamental difficulties in developing geometrical thought, as they can provide new representations of geometrical objects. Our aims are to evaluate how close from fulfilling this potential seem to be the instructional materials produced by the teachers themselves and to discuss the main factors influencing teachers' conceptions while developing of these materials.

INTRODUCTION

As discussed in Belfort & Guimarães (2002), the need to create opportunities for regular in-service training courses for secondary (that is: year groups from 11 to 17) Mathematics teachers in our geographical area led us to create a two-year in-service training course. We adopt as basic principles that solid mathematical formation and pedagogical and didactical knowledge are essential in training teachers, but they cannot be considered enough. The course should also include activities especially planned to allow teachers to reflect upon their classroom practices and to establish connections between their own learning experiences in Mathematics and their practice as secondary teachers.

During the first semester of the in-service training course, secondary Mathematics teachers revisit basic contents in geometry and functions, in disciplines supported by computer laboratory lessons. As reported in Guimarães et al. (2002), there are indications that most of teachers working in Brazilian secondary schools have *"a less than adequate grounding in geometry*" (p. 213), and it was decided that they should have several other opportunities to revisit geometry in the following semesters of the course.

They acquire some experience in using computers as tools for teaching Mathematics by attending the 'Computers for Mathematics Teaching' (CMT) discipline, which was designed to provide a counterpoint to the previous disciplines, when computers were extensively used as a tool by the instructors. With their newly acquired experience as learners to rely upon, they are asked in CMT to discuss, as well as exercise, the possibilities of the computer as a teaching tool for Mathematics. As part of the

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assessment process of the discipline, the teachers are asked to write an essay, in which they must present their own computational instructional materials to other secondary mathematics teachers and justify its use for classroom work. They were encouraged to use Dynamic Geometry (D.G.) software (Barbastefano et al., 2000; Jackiw, 1996; Laborde & Bellemain, 1994) as a way to help overcome their own difficulties in geometry.

A first report of the outcomes of a research project design to evaluate the CMT discipline was presented in Belfort et al. (2001). In this article, under the light of a broader theoretical perspective, we deepen that previous analysis and add new data from other editions of the discipline.

THEORETICAL BACKGROUND

There are several factors to consider if we endeavor to understand teacher's practices. Cooney (1988) described teaching as "an interactive process", in which conscious decision making is needed not only during the planning of the lessons but also "on the stage". Models to explain the role of the teacher usually consider the interaction between teacher's knowledge and beliefs as the basis for their decisions (Fennema et al., 1989). The complexity of the influence of textbooks and other written materials on teachers' practices is also to be considered (Belfort da Silva Moren, 2000).

There are also several indicators from research that a solid subject content knowledge may be essential for a successful teacher (see Grossman et al., 1989; Ball, 1988, 1991; Leinhardt et al., 1991). In particular, Ma (1999) discusses the importance of the "profound understanding of fundamental mathematics (PUFM)" demonstrated by some elementary teachers. According to her, the work of these teachers displays the following characteristics: connectedness, multiple perspectives, awareness of the basic ideas, and longitudinal coherence (p. 122).

If we attempt to transpose these research ideas to the situation of the study of Mathematics at secondary level, it seems that, if the teaching of secondary Mathematics is to display a similar set of characteristics as the ones observed by Ma (1999), the teacher would need to be able to ponder the connections of his/her mathematical knowledge with the mathematical contents of secondary school. In the case of Geometry, it seems that the secondary school curriculum oscillates between more figure exploration/less formal geometry teaching and less figures/more proof elaboration, reflecting the dialectic process between exploratory work with figures and proof elaboration, which can be seen in the historical evolution of geometry.

Concrete exploration, experimentation, conjecturing, problem solving and proof formulation seems to be central points of the set of skills we want the student to acquire in his/her knowledge construction process (Guimarães et al, 2002). But this set of skills, which seems so natural to the scientifically trained, does not come so naturally to the students. The concrete object does not have the same signification and is not explored in the same way by the mathematician and by the student: the way the concrete object is used strongly depends on the previous knowledge of who

is using it. Even more important is that teaching based on the exploration of the concrete object makes the none evident assumption that the interaction with the concrete will effectively produce the construction of the desired knowledge (Balacheff, 1999).

Again in the case of geometry, the concrete object is often a diagram, and to understand the differences in the exploration of this, researchers consider two different objects (Parzysz, 1988; Arsac, 1989; Laborde et Capponi, 1994; Balacheff, 1999): one that is concrete, called *the drawing*, and the other formal, corresponding to the class of drawings representing the same set of specifications (called *the figure*). From this point of view, a dynamic geometry software can have a specific contribution: it can provide controlled representations of geometrical objects, which, in some ways, concretize the formal figure.

We take as one of our assumptions that these software can provide new ways to learn geometry, and by way of consequence, new ways to teach geometry. Their use in in-service programs for teachers of Mathematics provides us also with the opportunity to discuss with them how to integrate mathematical software in their teaching toolkit.

ANALYSIS OF TEACHERS' INSTRUCTIONAL MATERIALS

In Belfort et al. (2001), we analyzed the instructional materials developed by the teachers during the first edition of CMT discipline from three different perspectives:

- usage of the computational software as a didactical resource (Software);
- subject matter knowledge and consistent mathematical reasoning (Subject); and
- appropriateness of the didactical proposal, considering the targeted year group (Instruction).

We discussed that, although we were expecting a balance among these three factors, this was achieved only in roughly one third of the essays. The majority was mostly oriented towards a single perspective, ignoring the others. During the following editions of the CMT discipline, the same tendencies were observed. We provide here some examples of the developed materials and briefly describe the outcomes of the discussions that took place during the presentations of these essays to the group.

Instructional Materials Oriented Towards the "Software" Perspective:

The instructional materials in this group (about 10% of the essays) reflect their authors' focus of interest as being the process of mastering the use of the software. The material developed by Marcos[1], using The Geometer's Sketchpad to draw geometric loci is an extreme example. Figure 1 illustrates two of his sketches.

The typical initial reaction by the teachers to this sort of presentation is admiring the sophisticated use of the package resources that characterizes these materials. Nevertheless, once the question "*what have you learned from this activity?*" is posed

[1] Here and in the following cases we use fictitious names.

to them, this reaction begins to change. In their final evaluation of the typical material in this group, teachers realize that they allow virtually no room for students' exploration and that mathematical concepts are neither explained nor justified. Teachers become aware that students are, at best, treated as spectators. For instance: in the case of Marcos' materials, all they have to do is to 'click the mouse' over the animation button, and watch curves being 'magically' traced, the proprieties of which they cannot investigate. It was also observed by the teachers the lack of definition of the year group(s) for which the material was intended.

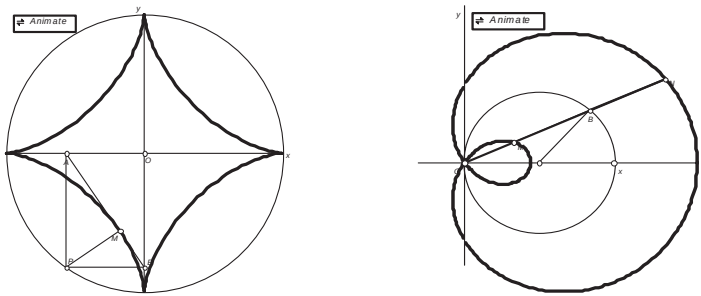

Figure 1: Marcos' instructional material in DG environment.

Instructional Materials Oriented Towards the "Subject" Perspective:

The instructional materials in this group (about 10% of the essays) usually reflect their authors' focus of interest as being the opportunity to revisit subject contents that can be represented in a simple way using DG. The work developed by José exemplifies this set of materials. As shown in figure 2, he used only the basic tools available in a Dynamic Geometry Software (Tabulæ) to study the points of intersection of the cevians of a triangle. The written essay was mathematically correct, presenting definitions, theorems and well-organized proofs.

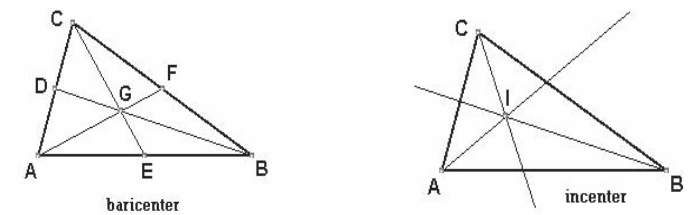

Figure 2: José's instructional materials in a DG environment

During the presentation of the works in this group, it soon became clear to the other teachers that very little is left for students' interactions, as these material are usually ready visualizations of mathematical results, not to be discussed nor justified in the laboratory environment. In the case of José's materials, for instance, the students' job

is reduced to move the vertices of the triangles to verify that the intersection points remain coincident. Since José could not state clearly the objectives of the proposed tasks, the group of teachers searched for didactical alternatives, and some interesting suggestions were presented. Teachers in the group were also unanimous in considering that students should be encouraged to construct the geometrical objects themselves, the more so given that only the simplest resources made available by the DG software were applied on the development of this type of sketches.

Instructional Materials Oriented Towards the "Instruction" Perspective:

Almost half of the materials developed by the teachers can be classed in this group. They reflect their authors' focus of interest in creating computer assisted learning activities for their pupils. Designed to provide experiences in area measurement, the instructional material presented by Mariana is a typical example. We present a model of the interaction proposed in her first sketch in figure 3: pupils are asked how many square area units are needed to fill the rectangle. The following sketches repeat the experience, but the rectangles get larger. Mariana explained to her colleagues that she expected the students to make the effort to (inductively) conclude the formula for the area of the rectangle, as a means of avoiding the repetitive job.

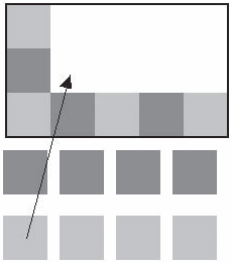

Figure 3: Mariana's instructional material in DG environment

Teachers' spontaneous remarks, such as "*this is so boring!*", " *after that, kids will hate computers!*" and "*this remind me of my textbook!*", show they acknowledged not only that the typical interaction proposed by these materials underestimates DG's potential but also the strong influence in these materials of textbook's approaches to Mathematics contents. That the proposed learning experiences in this group usually display a fragmented vision of the topic and a clear hurry in getting the "formula" is probably a consequence of this very influence.

Although these materials are often designed exploring the resources of DG software at an intermediate level, it was argued by the colleagues that they fail to encourage a meaningful mathematical development, lacking connections and multiple perspectives. For instance, in the particular case of Mariana's proposed activities, it was observed by one of the teachers that all rectangles in the sketches have integer side measures, even though the year group targeted by her is the very one studying operations with fractions in our schools. It is to be noted that this doesn't seem

relevant to many Brazilian textbook authors either, as a quick check on their own examples for area measurement indicates.

Well Balanced Instructional Materials:

An instructional material was considered as “well balanced” whenever all three perspectives were integrated by its author. The materials presented by Helena are a good example. Directed at students in their final years of secondary school, they exploit the characterization of the ellipse as the locus of the points for which the sum of the distances to two given fixed points of the plane is constant. By means of a sequence of activities supported by written guidelines, students are led to the construction of this locus using the so-called director circle. Figure 4 illustrates the initial stage of the third sketch given to the students and also its final stage, in which new constructions were added by the pupils. We have also observed that teachers who produce well balanced materials do not usually rely solely on the content of a single textbook, but seem to seek support for their work by researching a more comprehensive bibliography.

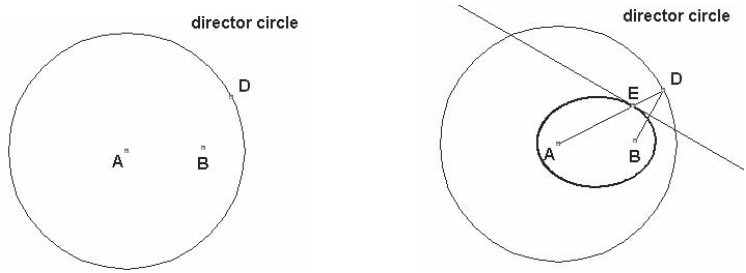

Figure 4: Helena’s instructional material in DG environment

Teachers who produced well balanced materials typically chose to simulate a laboratory lesson when presenting their work to their colleagues. During the presentation of these materials, it was observed that teachers, working in pairs, got really involved in solving the proposed tasks. The discussion engaged in by the group after these presentations acknowledged that examples in this category typically provide the students with a sequence of computer activities aiming at developing a well defined mathematical concept. Teachers in the group also commented that these materials display an evident concern with proper definitions and justifications for the geometric constructions and results.

FINAL REMARKS

The substantial differences in the outcomes of teachers’ work suggest that they have different views on Mathematics teaching and learning processes. Some of instructional materials seemed to reinforce the role of the teacher as the knowledge keeper (and teller), while others provided experiences that seemed to be designed to

keep students “busy”, with no clear (minimum) objectives to be achieved. Yet other materials seemed to reflect a problem identified by several researchers (Ball, 1988, Ma, 1999, etc.): the lack of deep, broad and thorough subject matter knowledge on the part of the teachers resulted in fragmented materials. In them, formulas were overestimated and there was strong reliance on the simplified approaches found in some of the poorest textbooks available in the Brazilian market. On the other hand, there were teachers who produced well balanced materials, in which mathematical concepts were treated as connected parts of a body of knowledge.

Our results also suggest that to provide their students with worthwhile learning experiences using computers is an idea that may have a strong appeal for the Mathematics teachers. We contend that this motivation can be explored to help them to overcome content knowledge difficulties, and to develop a critical awareness of the materials available for classroom work.

Although we are well aware that developing instructional materials is not at all a simple task (see Belfort da Silva Moren, 2000), we feel it is worthwhile to give teachers the opportunity to make an attempt at it. The debate that took place in classroom exploring the didactical characteristics of the materials made these teachers more conscious of some critical educational issues related to Mathematics teaching and learning processes, and, in most cases, made them willing to make the effort to overcome their perceived difficulties. It is to be expected that these experiences will reflect positively on their future work.

Finally, we strongly believe that a Dynamic Geometry software is a powerful tool for teaching and learning Mathematics. Nevertheless, as it happens with any other tool, it is the way it is used that determines the final outcomes. If we expect teachers to fully understand the potential of these packages, we’d better start to provide them with rich learning experiences supported by DG environments.

Acknowledgements

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